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DIGITAL FILTERS

by Edward B. Anders, et al.

Prepared under Contract No. NAS 8-5164 *by*
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for

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By Edward B. Anders, James J. Johnson, Alfred D. Lasaine,
Paul W. Spikes, and James T. Taylo

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ABSTRACT

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Digital (numerical) filtering techniques have become significant methods for data processing. This report presents the necessary background theory for a large class of digital filters. Particular filters discussed are those for smoothing, smoothing and first derivative, smoothing and second derivative, and smoothing and integration. Error bounds are given for the first three types.

Author

PREFACE

Under Contract No. NAS8-5164 with NASA, Auburn Research Foundation undertook a program to study numerical smoothing and differentiation methods and numerical aspects of finite difference methods. The work was performed by the Mathematics Department, Auburn University.* The Senior Investigators were Dr. Nathaniel Macon, 1 October 1962 to 1 June 1963, and Dr. L. P. Burton, 1 June 1963 to 31 July 1964. Investigators were Edward B. Anders, 1 October 1962 to 1 June 1963, Paul W. Spikes, 1 October 1962 to 1 September 1963, Alfred D. Lasaine, 1 October 1962 to 1 June 1964, James T. Taylo, 1 June 1963 to 20 July 1964, and James J. Johnson, 1 September 1963 to 1 June 1964.

The main effort was devoted to linear digital (numerical) filters for performing the smoothing, differentiation, and integration of discrete data and to error analysis for these filters. The primary interest was filtering techniques for one variable, but an extension to n variables was developed by Anders [14].

This report incorporates the necessary background theory starting with the classical Fourier theory and going into generalized functions, pertinent results obtained by other workers, and the results and conclusions obtained under this contract. Numerous references to other publications are given.

A reader interested only in the application of the techniques discussed here may start reading with Chapter IV.

E.B.A.
J.J.J.
A.D.L.
P.W.S.
J.T.T.

* With Ronald J. Graham and David G. Aichele as NASA Technical Representatives.

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NOMENCLATURE

t: time

f: frequency

$\delta(t)$: Dirac delta function

$f(t)$: a function f of t

$h(t)$: weight function of a smoothing filter

$H(f)$: transfer function of a smoothing filter

$y^n(t)$: n^{th} derivative weight function, $n \geq 1$.

$Y^n(f)$: n^{th} derivative transfer function, $n \geq 1$.

$b^n(t)$: n^{th} derivative band-pass weight function, $n \geq 0$.

$B^n(f)$: n^{th} derivative band-pass transfer function, $n \geq 0$.

$y^{(-1)}(t)$: integrator weight function

$Y^{(-1)}(f)$: integrator transfer function

f_s: sampling rate

Δt: $\frac{1}{f_s}$

f_α: highest frequency present in signal

f_c: cut-off frequency

f_T: termination frequency

Δf: roll-off length

h_k, y_k^n, etc.: k^{th} filter weights

w: angular frequency ($w=2\pi f$)

r: frequency ratio $\frac{f}{f_s} = \frac{w}{w_s}$

r_c: cut-off ratio $\frac{f_c}{f_s}$

r_T: termination ratio $\frac{f_T}{f_s}$

r_d: roll-off length ratio $\frac{\Delta f}{f_s}$

N: number of weights on each side of the center weight h_o.

$\doteq$: approximately equal to

Si(x): sine integral

α, Δ, δ: constraint terms or factors

CHAPTER I

CLASSICAL FOURIER ANALYSIS

In this chapter we shall list, without proof, some of the results from Classical Fourier Analysis. Proofs of these results are readily available in many good texts on the subject [1], [2].

Fourier integral theorems

Given a function $f(x)$ of a real variable x, if $f(x)$ is absolutely integrable, i.e., if

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty, \tag{1.1}$$

then the integral

$$F(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\,dx \tag{1.2}$$

exists and

$$f(x) = \int_{-\infty}^{\infty} F(y) e^{2\pi ixy}\,dy. \tag{1.3}$$

The condition (1.1) is sufficient but not necessary for (1.2) and (1.3). We assume that all functions with which we shall be concerned in this chapter satisfy conditions necessary for the existence of (1.2) and (1.3).

The usual notation

$$f(x) \longleftrightarrow F(y)$$

will be used to indicate that the functions of $f(x)$ and $F(y)$ are related by (1.2) and (1.3). We call $F(y)$ the Fourier transform of $f(x)$, and $f(x)$ the inverse Fourier transform of $F(y)$. This being the only type of transform we shall use, it is occasionally convenient to drop the word "Fourier" and speak of the "transform of $f(x)$" and of the "inverse transform of $F(y)$."

F(y) is in general complex:

$$F(y) = R(y) + iI(y) \tag{1.4a}$$

or

$$F(y) = A(y)e^{i\theta(y)} \tag{1.4b}$$

A(y) is called the Fourier spectrum of f(x), $A^2(y)$ is its energy spectrum, and $\theta(y)$ its phase angle.

The function f(x) may in general be complex:

$$f(x) = f_R(x) + if_I(x)$$

and

$$F(y) = \int_{-\infty}^{\infty} [f_R(x) + if_I(x)][\cos 2\pi xy - i \sin 2\pi xy]dx$$

$$= \int_{-\infty}^{\infty} [f_R(x) \cos 2\pi xy + f_I(x) \sin 2\pi xy]dx$$

$$-i\int_{-\infty}^{\infty} [f_R(x) \sin 2\pi xy - f_I(x) \cos 2\pi xy]dx. \tag{1.5}$$

Hence

$$R(y) = \int_{-\infty}^{\infty} [f_R(x) \cos 2\pi xy + f_I(x) \sin 2\pi xy]dx \tag{1.6}$$

and

$$I(y) = -\int_{-\infty}^{\infty} [f_R(x) \sin 2\pi xy - f_I(x) \cos 2\pi xy]dx. \tag{1.7}$$

Using (1.4a) we find, in a similar manner, that

$$f_R(x) = \int_{-\infty}^{\infty} [R(y) \cos 2\pi xy - I(y) \sin 2\pi xy]dy. \tag{1.8}$$

and

$$f_I(x) = \int_{-\infty}^{\infty} [R(y) \sin 2\pi xy + I(y) \cos 2\pi xy]dy. \tag{1.9}$$

Suppose that f(x) is a real function of x. Then

$$R(y) = \int_{-\infty}^{\infty} f(x) \cos 2\pi xy dx$$

and (1.10)

$$I(y) = \int_{-\infty}^{\infty} f(x) \sin 2\pi xy dx.$$

It is easily seen that

$$R(-y) = R(y)$$

and (1.11)

$$I(-y) = -I(y),$$

i.e., R(y) is even and I(y) is odd.

Hence

$$F(-y) = F^*(y) \tag{1.12}$$

where the asterisk indicates the complex conjugate.

Conversely, if (1.12) holds, then

$$f_I(x) = \int_{-\infty}^{\infty} [R(y) \sin 2\pi xy + I(y) \cos 2\pi xy] dy$$

$$= 0$$

because the integral is odd, and therefore f(x) is real.

Suppose that f(x) is an imaginary function of x. Then

$$R(y) = \int_{-\infty}^{\infty} f(x) \sin 2\pi xy dx$$

and (1.13)

$$I(y) = \int_{-\infty}^{\infty} f(x) \cos 2\pi xy dx.$$

Hence R(y) is odd and I(y) is even, and

$$F(-y) = -F^*(y). \tag{1.14}$$

Conversely, if (1.14) holds, then f(x) is imaginary.

Even and odd functions

If $F(y)$ is even, then

$$f(x) = 2\int_0^{\infty} F(y) \cos 2\pi xy dy. \tag{1.15}$$

And if $F(y)$ is odd, then

$$f(x) = 2i\int_0^{\infty} F(y) \sin 2\pi xy dy. \tag{1.16}$$

Similar expressions are found for $F(y)$, $f(x)$ being even or odd.

We will now list some of the more important theorems of classical Fourier analysis.

I. Linearity

If $F(y)$ and $G(y)$ are the transforms of $f(x)$ and $g(x)$ respectively, and if a,b are arbitrary constants, then

$$a\,F(y) + b\,G(y) \longleftrightarrow a\,f(x) + b\,g(x). \tag{1.17}$$

II. Symmetry

If $F(y)$ is the transform of $f(x)$, then

$$F(x) \longleftrightarrow f(-y). \tag{1.18}$$

III. If a is a non-zero real constant and $F(y)$ is the transform of $f(x)$, then

$$f(ax) \longleftrightarrow \frac{1}{|a|}\, F\left(\frac{y}{a}\right). \tag{1.19}$$

IV. "x" domain shifting

If x_o is a real constant and $F(y)$ is the transform of $f(x)$, then

$$f(x-x_o) \longleftrightarrow F(y)\, e^{-2\pi i x_o y} \tag{1.20}$$

V. "y" domain shifting

If y_o is a real constant and $F(y)$ is the transform of $f(x)$, then

$$e^{2\pi i x y_o}\, f(x) \longleftrightarrow F(y-y_o) \tag{1.21}$$

Note: Using (1.19) and (1.21) we have

$$e^{2\pi i x y_0} f(ax) \longleftrightarrow \frac{1}{|a|} F\left(\frac{y-y_0}{a}\right) \qquad (1.22)$$

VI. <u>"x" domain differentiation</u>

If the transform of $\frac{d^n f(x)}{dx^n}$ exists and if $F(y)$ is the transform of $f(x)$, then

$$\frac{d^n f(x)}{dx^n} \longleftrightarrow (2\pi i y)^n F(y) \qquad (1.23)$$

VII. <u>"y" domain differentiation</u>

If the inverse transform of $\frac{d^n F(y)}{dy^n}$ exists and if $F(y)$ is the transform of $f(x)$, then.

$$(-2\pi i x)^n f(x) \longleftrightarrow \frac{d^n F(y)}{dy^n} \qquad (1.24)$$

VIII. <u>Conjugate functions</u>

If $F(y)$ is the transform of $f(x)$ and the asterisk indicates the conjugate, then

$$f^*(x) \longleftrightarrow F^*(-y). \qquad (1.25)$$

IX. <u>"x" domain convolution</u>

If $F(y)$ is the transform of $f(x)$ and $G(y)$ is the transform of $g(x)$, then

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(z)g(x-z)\,dz \longleftrightarrow F(y)G(y) \qquad (1.26)$$

We note here that the existence of the transforms of $f(x)$ and $g(x)$ is not sufficient to prove this theorem. The proof usually goes as follows:

Let
$$H(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} \left[\int_{-\infty}^{\infty} f(z)g(x-z)\,dz\right] dx.$$

Assuming that the order of integration can be changed, we have

$$H(y) = \int_{-\infty}^{\infty} f(z) \left[\int_{-\infty}^{\infty} e^{-2\pi ixy} g(x-z)\,dx \right] dz,$$

Using (1.20) we have

$$H(y) = \int_{-\infty}^{\infty} f(z) e^{-2\pi iyz} G(y)\,dz$$

$$= F(y)G(y).$$

A sufficient condition which would allow the interchange in the order of integration is that $f(x)$ and $g(x)$ be square-integrable, i.e.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

and

$$\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty. \tag{1.27}$$

Similarly, we have the following theorem which can be proved from (1.26) by using (1.18).

X. <u>"y" domain convolution</u>

If $F(y)$ is the transform of $f(x)$ and $G(y)$ is the transform of $g(x)$, then

$$f(x)g(x) \longleftrightarrow \int_{-\infty}^{\infty} F(z)G(y-z)\,dz = F(y)*G(y). \tag{1.28}$$

XI. <u>Parseval's formula</u>

If $F(y)$ and $G(y)$ are the transforms of $f(x)$ and $g(x)$ respectively, then

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} F(-y)G(y)\,dy. \tag{1.29}$$

CHAPTER II

GENERALIZED FUNCTIONS AND THEIR FOURIER TRANSFORMS

2.1 Generalized functions

The Dirac delta function $\delta(t)$ is often "defined" by one of the following statements:

(A) If $f(t)$ is a continuous function at $t=t_o$, then $\delta(t)$ has the property that

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_o)\,dt = f(t_o); \tag{2.1}$$

(B) $\delta(t) = o$ if $t \neq o$, and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1; \tag{2.2}$$

(C) $\delta(t) = \lim_{n\to\infty} f_n(t)$ where $\{f_n(t)\}$ is a sequence of functions satisfying the conditions:

$$\int_{-\infty}^{\infty} f_n(t)\,dt = 1 \quad \text{and} \quad \lim_{n\to\infty} f_n(t) = o,\ t \neq o. \tag{2.3}$$

These definitions are meaningless if we attempt to think of $\delta(t)$ as a function in the ordinary sense. By introducing the delta function as a new concept, a generalized function, the difficulties are resolved. Then (2.1) can be given precise meaning, but (2.2) and (2.3) do not uniquely describe $\delta(t)$. Such an approach to the problem is usually relegated to an appendix of a book, and is usually very sketchy.

We are concerned here with extending the Fourier integral theory. A relatively easy and short approach, which is similar to the extension of the rational numbers to the real numbers, is given by Lighthill [3].

Alternately, we could delve into functional analysis to extend the theory. For this study it appears that the first approach will be more meaningful, and so we have chosen to develop it. However we shall restrict ourselves to that portion of the development which suits our purpose here, and for a detailed development the reader is referred to Lighthill.

Advantages in clarity are gained by using some of the terminology and notation of functional analysis.

Let R be a set of scalars (the real or complex numbers) and let K be a set with an operation denoted by (+) and called "addition" defined on it.

Definition 1. K is called a linear space over R if

1) K is an abelian group with respect to (+)
2) $ak \in K$ for all $a \in R$ and all $k \in K$, and
   (a) $a(bk) = (ab)k$ for all $a,b \in R$,
   (b) $1 \cdot k = k$
3) If $a,b \in R$ and $k_1, k_2 \in K$, then
   (a) $a(k_1+k_2) = ak_1+ak_2$
   (b) $(a+b)k_1 = ak_1+bk_1$.

Let

$$X = \{x \mid x \text{ is a real number}\}$$

and

$$H = \{h \mid h \text{ is a function of } x,\ x \text{ a real number}\}.$$

Under ordinary addition and scalar multiplication, H is a linear space over X.

We denote the Fourier transform of f(x) by Z(f),

$$Z(f) = g(y) = \int_{-\infty}^{\infty} f(x)c^{-2\pi ixy}dx.$$

We denote the inverse transform by $Z^{-1}(g) = f$. Then Z is a mapping of a subset Z_x of H into H, and Z^{-1} is a mapping of a subset Z_x^{-1}

of H into H. However $Z_x^{-1} \neq Z_x$, that is, the domains of definition of Z and Z^{-1} in H are not identical. In addition, there exist convergent sequences of functions of Z_x, each element of which has a transform, but whose limit has no transform.

Z_x and Z_x^{-1} are linear spaces. This suggests the possibility of constructing a linear space in which these may be imbedded and such that the mappings Z and Z^{-1} can be suitably extended to remove the above difficulties.

Let S denote the set of all functions f(x) which are everywhere differentiable any number of times and such that f(x) and all of its derivatives are $O(|x|^{-N})$ as $|x| \longrightarrow \infty$ for all integers N. As a reminder, the O notation, $f(x) = O(g(x))$ as $x \longrightarrow a$, means that there exists a positive constant A such that

$$|f(x)| < A|g(x)|$$

as $x \longrightarrow a$.

Example: e^{-x^2} is contained in S.

It is clear that S, under ordinary addition and scalar multiplication of functions, is a linear space. S is called a <u>test</u> or <u>fundamental</u> space and elements of S are called <u>test functions</u>. If g(x) is a function such that $g(x)f(x) \in S$ for all $f(x) \in S$, then g(x) is called a <u>multiplier</u> on S.

Let M denote the set of all functions m(x) which are everywhere differentiable any number of times and such that m(x) and all of its derivatives are $O(|x|^{N_o})$ as $|x| \longrightarrow \infty$ for some integer N_o.

<u>Lemma</u>: If $m(x) \in M$ and $f(x) \in S$, then $m(x)f(x) \in S$.

<u>Proof</u>: $$\frac{d^p}{dx^p}\left\{m(x)f(x)\right\} = \sum_{j=0}^{p} \binom{p}{j} m^{(j)}(x) f^{(p-j)}(x).$$

It suffices to show each term on the right in the above equation is in S. There exist numbers $A_1 > 0$, $K_1 > 0$ and an integer N_1 such that

$$|m^{(j)}(x)| \leq A_1 |x|^{N_1} \quad \text{for all } x \text{ such that } |x| > K_1.$$

If N_2 is any integer, then there exist numbers $A_2 > 0$, $K_2 > 0$ such that

$$|f^{(p-j)}(x)| \leq A_2 |x|^{-N_2} \text{ for all } x \text{ such that } |x| > K_2.$$

Then for all x such that $|x| > \max [K_1, K_2]$,

$$|m^{(j)}(x) f^{(p-j)}(x)| \leq A_1 A_2 |x|^{N_1 - N_2}.$$

But $N = N_1 - N_2$ is arbitrary since N_2 is arbitrary. Hence $m^{(j)}(x)$

$f^{(p-j)}(x) \in S$. This shows that the elements of M are multipliers on S. Obviously if $m(x) \in M$, then $m'(x)$ and $m'(x)f(x) \in S$.

Example: Any polynomial is contained in M.

Theorem 1. If $f(x) \in S$, then

(a) $f'(x) \in S$

(b) $Z(f) = g(y) \in S$

(c) $Z^{-1}(f) = h(y) \in S$

(d) $f(-x) \in S$

(e) $f^*(x) \in S$

(f) $f(ax+b) \in S$, a,b constants and $a \neq 0$.

Proof: The first part is obvious. To prove (b), let

$$g(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi ixy} dx$$

and differentiate p times. Then integrate by parts N times. Thus

$$\left|g^{(p)}(y)\right| = \left| \frac{1}{(2\pi iy)^N} \int_{-\infty}^{\infty} \frac{d^N}{dx^N} \left\{(-2\pi ix)^p f(x)\right\} e^{-2\pi ixy} dx \right|$$

$$\leq \frac{(2\pi)^{p-N}}{|y|^N} \int_{-\infty}^{\infty} \left| \frac{d^N}{dx^N} \left\{x^p f(x)\right\} \right| dx$$

Therefore

$$g^{(p)}(y) = O(|y|^{-N}).$$

Part (c) is proved in a similar manner. For part (d), from (b) we have $g(y) = Z(f)$ is a test function. Applying Z^{-1} we have

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}dy.$$

Hence

$$f(-x) = \int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}dy,$$

i.e., $f(-x) = Z(g)$, hence $f(-x) \in S$.

Part (e) follows by virtue of the fact that $|f^{(m)}(x)| = |f^{*(m)}(x)|$.

For part (d), let $f(x) \longleftrightarrow g(y)$ and use (1.19) and (1.20) to write

$$f(ax+b) \longleftrightarrow \frac{e^{2\pi i\frac{by}{a}}}{|a|} g(\tfrac{y}{a}). \tag{2.5}$$

Then it suffices to show that the right side of (2.5) is a test function. Clearly it is everywhere differentiable any number of times. Since

$$\left|\frac{e^{2\pi i\frac{by}{a}}}{a} g(\tfrac{y}{a})\right| \leq \left|\frac{g(\frac{y}{a})}{a}\right|,$$

it is obviously $O(|y|^{-N})$ as $|y| \longrightarrow \infty$ for all integers N. Hence $f(ax+b) \in S$.

<u>Definition 2</u>. A sequence $\{f_n(x)\}$ of test functions is called <u>regular</u> if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)F(x)dx < \infty \text{ for all } F(x) \in S. \tag{2.6}$$

An example of such sequence is $\left\{e^{\frac{-x^2}{n^2}}\right\}$.

<u>Definition 3</u>. Let C denote the class of all regular sequences. Then $\{f_n(x)\}$ $\in$ C is said to be <u>equivalent</u> to $\{g_n(x)\}$ $\in$ C if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(x)F(x)\,dx \text{ for all } F(x) \in S. \tag{2.7}$$

It is easy to see that this is an equivalence relation. Hence C is partitioned into disjoint subclasses, and we let $\bar{S}$ denote the set of all subclasses of C determined by this equivalence relation.

Definition 4. An element of $\bar{S}$ is called a generalized function.

Hence a generalized function is the class of all regular sequences equivalent to a given regular sequence. Since the limit (2.6) is the same for all sequences of a given element s of $\bar{S}$, any sequence of s can serve as a representative of that class.

Definition 5. If $s(x) \in \bar{S}$ and $F(x) \in S$ we define

$$\int_{-\infty}^{\infty} s(x)F(x)\,dx \equiv \lim_{n\to\infty}\int_{-\infty}^{\infty} s_n(x)F(x)\,dx \tag{2.8}$$

where $\{s_n(x)\}$ is any representative of $s(x)$.

In general, the left side of (2.8) has no meaning as an integral in the ordinary sense. It is just a conventional notation we adopt to denote the limit on the right side of (2.8).

Example: The sequence $\left\{e^{\frac{-x^2}{n^2}}\right\}$ represents a generalized function $I(x)$ such that

$$\int_{-\infty}^{\infty} I(x)F(x)\,dx = \int_{-\infty}^{\infty} F(x)\,dx. \tag{2.9}$$

Definition 6. Let $f(x)$ and $h(x)$ be generalized functions with representative sequences $\{f_n(x)\}$ and $\{h_n(x)\}$ respectively. Also let $m(x) \in M$. Then

(a) the sum $f(x) + h(x)$ is defined as the generalized function $b(x)$ with a representative $\{f_n(x) + h_n(x)\}$;

(b) the derivative $f'(x)$ is defined as the generalized function with a representative $\{f_n'(x)\}$;

(c) $f(ax+b)$ is defined as the generalized function with a representative $\{f_n(ax+b)\}$;

(d) The product $m(x)f(x)$ is defined as the generalized function with a representative $\{m(x)f_n(x)\}$;

(e) The Fourier transform $g(y)$ of $f(x)$ is defined as the generalized function with a representative $\{g_n(y)\}$, where $g_n(y)$ is the Fourier transform of $f_n(x)$.

To show that these definitions are consistent we must show that

(a) each sequence named is a sequence of test functions,

(b) each sequence named is regular,

and

(c) equivalent regular sequences defining $f(x)$ and $h(x)$ lead to equivalent regular sequences defining the new generalized functions.

Part (a) follows in each case from previous remarks and Theorem 1. Parts (b) and (c) can be deduced in each case from the following equations:

1) $$\lim_{n\to\infty}\int_{-\infty}^{\infty}[f_n(x)+h_n(x)]F(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}f_n(x)F(x)\,dx + \lim_{n\to\infty}\int_{-\infty}^{\infty}h_n(x)F(x)\,dx, \tag{2.10}$$

2) $$\lim_{n\to\infty}\int_{-\infty}^{\infty}f_n'(x)F(x)\,dx = -\lim_{n\to\infty}\int_{-\infty}^{\infty}f_n(x)F'(x)\,dx, \tag{2.11}$$

3) $$\lim_{n\to}\int_{-\infty}^{\infty}f_n(ax+b)F(x)\,dx = \frac{1}{|a|}\lim_{n\to\infty}\int_{-\infty}^{\infty}f_n(x)F\left(\frac{x-b}{a}\right)dx \tag{2.12}$$

4) $$\lim_{n\to\infty}\int_{-\infty}^{\infty}[m(x)f_n(x)]F(x)\,dx = \lim_{n\to}\int_{-\infty}^{\infty}f_n(x)[m(x)F(x)]\,dx \tag{2.13}$$

and, from Parseval's formula,

5) $$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(y)G(y)\,dy = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(-x)\,dx, \qquad (2.14)$$

where $G(y)$ is the Fourier transform of $F(x)$.

It follows from the equations (2.10)--(2.14) and definition 5 that

$$\int_{-\infty}^{\infty} f'(x)F(x)\,dx = -\int_{-\infty}^{\infty} f(x)F'(x)\,dx, \qquad (2.15)$$

$$\int_{-\infty}^{\infty} f(ax+b)F(x)\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)F\left(\frac{x-b}{a}\right)dx, \qquad (2.16)$$

$$\int_{-\infty}^{\infty} [m(x)f(x)]F(x)\,dx = \int_{-\infty}^{\infty} f(x)[m(x)F(x)]\,dx, \qquad (2.17)$$

and

$$\int_{-\infty}^{\infty} h(y)G(y)\,dy = \int_{-\infty}^{\infty} f(x)F(-x)\,dx \qquad (2.18)$$

If a is a scalar, then the function $h(x) = a$ is contained in M and hence if $f(x) \in \bar{S}$, $af(x) \in \bar{S}$. With the addition as defined above in $\bar{S}$, we have $\bar{S}$ is a linear space.

Let I denote the set of all ordinary functions $h(x)$ such that $(1+x^2)^{-N}h(x)$ is absolutely integrable on $(-\infty, \infty)$ for some integer N.

Theorem 2. I is a linear space and can be imbedded in $\bar{S}$, i.e., $I \subset \bar{S}$.

The theorem asserts that if $h(x) \in I$ then there exists a regular sequence $\{h_n(x)\}$ such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(x)F(x)\,dx = \int_{-\infty}^{\infty} h(x)F(x)\,dx, \text{ for all } F(x) \in S. \qquad (2.19)$$

For a proof of this theorem we refer the reader to [3], pp. 22-23.

Having shown the existence of such a sequence, the mapping

$$h \rightarrow C\left\{h_n\right\},$$

where $C\left\{h_n\right\}$ denotes the class of all sequences equivalent to $\left\{h_n\right\}$, imbeds I in $\bar{S}$.

Note that the integral on the right side of (2.19) is

$$\int_{-\infty}^{\infty} h(x)F(x)\,dx = \int_{-\infty}^{\infty} \left\{(1+x^2)^{-N}h(x)\right\} \quad \left\{(1+x^2)^{N}F(x)\right\} dx.$$

Since $(1+x^2)^{-N}h(x)$ is absolutely integrable and $(1+x^2)^{N}F(x)$ is a test function, this integral exists in the ordinary sense.

<u>Definition 7</u>. If $h(x) \in I$, then the image of $h(x)$ in $\bar{S}$ is called the <u>generalized function defined by h(x)</u> and is denoted by the same symbol.

Thus each ordinary function $h(x) \in I$ defines a generalized function and the integral (2.19) has meaning in the ordinary sense and in the generalized theory, and in this case these meanings are the same.

Using (1.18), (1.23) and (2.5), with definition 6 we deduce:

<u>Theorem 3</u>. If $f(x) \in \bar{S}$ and $Z(f) = g(y)$, then

(a) $Z[f(ax+b)] = \dfrac{e^{2\pi i\frac{by}{a}}}{|a|}\, g(\frac{y}{a}),\ a \neq 0,$

(b) $Z[f'(x)] = 2\pi i y g(y),$

and

(c) $Z[g(-x)] = f(y).$

The following two theorems eliminate the possibility of confusion in the notations $f(x)$ and $f'(x)$, where $f(x)$ and $f'(x)$ are contained in I.

<u>Theorem 4</u>. If $f(x)$ and $f'(x)$ exist as ordinary functions contained in the set I, then the derivative of the generalized function defined

by $f(x)$ is the generalized function defined by $f'(x)$.

Proof: In the generalized theory, with $f(x)$ and $f'(x)$ interpreted as generalized functions, we have shown that

$$\int_{-\infty}^{\infty} f'(x)F(x)\,dx = -\int_{-\infty}^{\infty} f(x)F'(x)\,dx, \tag{2.20}$$

for all test functions $F(x)$.

Considering $f(x)$ and $f'(x)$ as ordinary functions the integrals

$$\int_{-\infty}^{\infty} f'(x)F(x)\,dx,$$

$$\int_{-\infty}^{\infty} f(x)F(x)\,dx,$$

and

$$\int_{-\infty}^{\infty} f(x)F'(x)\,dx$$

each exist. Integrating the first integral by parts we have

$$\int_{-\infty}^{\infty} f'(x)F(x)\,dx = \lim_{\substack{a\to\infty \\ b\to\infty}} f(x)F(x)\Big|_{-a}^{b} - \int_{-\infty}^{\infty} f(x)F'(x)\,dx.$$

Hence

$$\lim_{x\to -\infty} f(x)F(x)$$

and

$$\lim_{x\to\infty} f(x)F(x)$$

must be finite. But the existence of $\int_{-\infty}^{\infty} f(x)F(x)\,dx$ implies that both limits are zero. Hence (2.20) holds for ordinary functions $f(x)$ and $f'(x)$.

Theorem 5. If $f(x)$ is an ordinary function which is absolutely integral on $(-\infty, \infty)$--so that its Fourier transform $g(y)$ exists

by the classical Fourier integral theorem--then the Fourier transform of the generalized function $f(x)$ is the generalized function $g(y)$.

Proof: We have

$$\int_{-\infty}^{\infty} |[1+y^2]^{-1} g(y)| dy = \int_{-\infty}^{\infty} |[1+y^2]^{-1} \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} dx| dy$$

$$\leq [\int_{-\infty}^{\infty} [1+y^2]^{-1} dy] \int_{-\infty}^{\infty} |f(x)| dx$$

$$< \infty.$$

Hence $g(y)$ satisfies definition 7, and the generalized function $g(y)$ exists. Now both the ordinary function $g(y)$ and the generalized function $g(y)$ satisfy

$$\int_{-\infty}^{\infty} g(y)G(y)dy = \int_{-\infty}^{\infty} f(x)F(-x)dx$$

for any test function $F(x)$, where $Z[F] = G(y)$. Therefore, the Fourier transform of the generalized function $f(x)$ is the one defined by $g(y)$.

2.2 The Dirac delta function

1. The sequence $\left\{(\frac{n}{\pi})^{\frac{1}{2}} e^{-nx^2}\right\}$ represents the important Dirac delta function $\delta(x)$ which has the property that for any test function $F(x)$,

$$\int_{-\infty}^{\infty} \delta(x)F(x)dx = F(0). \tag{2.21}$$

To show this, we note that

(a) $\int_{-\infty}^{\infty} (\frac{n}{\pi})^{\frac{1}{2}} e^{-nx^2} dx = 1;\ n = 1, 2, \ldots$

(b) $\int_{-\infty}^{\infty}(\frac{n}{\pi})^{\frac{1}{2}}|x|e^{-nx^2}\,dx = (\frac{1}{n\pi})^{\frac{1}{2}}.$

If $F(x) \in S$, using (a) we write

$$\left|\int_{-\infty}^{\infty}(\frac{n}{\pi})^{\frac{1}{2}}e^{-nx^2}F(x)dx-F(0)\right| = \left|\int_{-\infty}^{\infty}(\frac{n}{\pi})^{\frac{1}{2}}e^{-nx^2}[F(x)-F(0)]dx\right|$$

$$= \left|\int_{-\infty}^{\infty}(\frac{n}{\pi})^{\frac{1}{2}}e^{-nx^2}\frac{[F(x)-F(0)]}{x}\,xdx\right|$$

$$\leq \int_{-\infty}^{\infty}(\frac{n}{\pi})^{\frac{1}{2}}|x|e^{-nx^2}\left|\frac{F(x)-F(0)}{x}\right|dx$$

$$\leq \frac{\max|F'(x)|}{(\pi n)^{\frac{1}{2}}} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

where in the last line we used the Mean Value theorem for derivatives, i.e., on any interval $[0,x]$ there exists an a, $0 < a < x$, such that $F'(a) = \frac{F(x)-F(0)}{x}$ and thus $\max|F'(x)| \geq |F'(a)|$ on any interval $[0,x]$; and (b).

2. The Fourier transform of $\delta(x)$ is 1. To see this we note that

$$(\frac{n}{\pi})^{\frac{1}{2}}e^{-nx^2} \longleftrightarrow e^{\frac{-\pi^2y^2}{n}},$$

and hence the sequence $\left\{e^{\frac{-\pi^2y^2}{n}}\right\}$ is a representative of the Fourier transform of $\delta(x)$. But for any test function $G(y)$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{\frac{-\pi^2y^2}{n}}G(y)dy = \int_{-\infty}^{\infty}1\cdot G(y)dy,$$

hence

$$\delta(x) \longleftrightarrow 1.$$

3. The Fourier transform of $\delta(x-x_o)$ is $e^{-2\pi i x_o y}$. This follows easily from part (a) of theorem 3.

4. From $\delta(x-x_o) \longleftrightarrow e^{-2\pi i x_o y}$ and part (c) of theorem 3, we have

$$e^{2\pi i x y_o} \longleftrightarrow \delta(y-y_o).$$

For $y_o = 0$, we have

$$1 \longleftrightarrow \delta(y).$$

5. Writing $\cos 2\pi x y_o = \frac{1}{2}[e^{2\pi i x y_o} + e^{-2\pi i x y_o}]$ and applying the above relation gives

$$\cos 2\pi x y_o \longleftrightarrow \frac{1}{2}[\ \delta(y-y_o) + \delta(y+y_o)].$$

Writing $\sin 2\pi x y_o = \frac{1}{2i}[e^{2\pi i x y_o} - e^{-2\pi i x y_o}]$ and using the same relationship, we find that

$$\sin 2\pi x y_o \longleftrightarrow \frac{1}{2i}[\ \delta(y-y_o) - \delta(y+y_o)].$$

Definition 8. Let g(x) be an ordinary function such that, for any test function F(x) which is zero outside of (a,b), g(x)F(x) is integrable on (a,b), $a \leq b$. If f(x) is a generalized function such that

$$\int_{-\infty}^{\infty} f(x)F(x)dx = \int_{-\infty}^{\infty} g(x)F(x)dx \qquad (2.22)$$

then we define

$$f(x) = g(x),$$

for $a < x < b$.

In the sense of this definition we have $\delta(x) = 0$ for $0 < x < \infty$ and $-\infty < x < 0$. Suppose $F(x) = 0$ when $0 < x < \infty$ and when $-\infty < x < 0$,

then by continuity $F(0) = 0$ and

$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = 0.$$

2.3 The Convolution theorem.

In section 2.1, we have shown that (1.17) - (1.22) hold for generalized functions. It can be shown that (1.23) and (1.24) also hold. However, in order to extend (1.26) and (1.28) to include generalized functions, some restrictions must be placed on these functions. If we try to give the symbol

$$(f * g)(x) = \int_{-\infty}^{\infty} f(z)g(x-z)\,dz \tag{2.23}$$

a meaning for generalized functions $f(x)$ and $g(x)$, then in the case where $f(x)$ and $g(x)$ are ordinary functions, (2.23) must have the same meaning as ordinary convolution. We have already found that for the convolution theorem restrictions are necessary on the ordinary functions $f(x)$ and $g(x)$.

Suppose the generalized functions $f(x)$ and $g(x)$ are defined by the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ respectively. Consider the sequence

$$a_n(x) = \int_{-\infty}^{\infty} f_n(z)g_n(x-z)\,dz. \tag{2.24}$$

For each n, $a_n(x)$ is a test function. To see this, we let $f_n(x) \longleftrightarrow F_n(y)$ and $g_n(x) \longleftrightarrow G_n(y)$. Applying the classical convolution theorem gives

$$a_n(x) = \int_{-\infty}^{\infty} F_n(y)G_n(y)e^{2\pi ixy}\,dy.$$

Now $F_n(y)G_n(y)$ is a test function and, by theorem 1, so is $a_n(x)$.

For $H(x) \in S$ with $Z^{-1}(H) = h(y)$

$$\int_{-\infty}^{\infty} a_n(x)H(x)dx = \int_{-\infty}^{\infty} H(x)dx \int_{-\infty}^{\infty} F_n(y)G_n(y)e^{2\pi ixy}dy$$

$$= \int_{-\infty}^{\infty} F_n(y)G_n(y)dy \int_{-\infty}^{\infty} H(x)e^{2\pi ixy}dx$$

$$= \int_{-\infty}^{\infty} F_n(y)G_n(y)h(y)dy.$$

Now $h(y)$ is a test function, but in general the sequence $\{F_n(y)G_n(y)\}$ is not regular. Hence $\{a_n(x)\}$ may not define a generalized function.

In the functional analysis approach (see [4], p. 106), it is shown that by appropriately restricting $f(x)$, $(f*g)(x)$ is a generalized function for all $g(x) \in \bar{S}$. Of primary interest to us here is the case when $f(x)$ is a linear combination of Dirac delta functions. To avoid some tedious convergence problems which arise in a general approach, we shall restrict our discussion to this case.

Let $g(x)$ be any generalized function defined by $\{g_n(x)\}$ and define $\delta(x) * g(x)$ by the sequence

$$a_n(x) = \int_{-\infty}^{\infty} \delta(z)g_n(x-z)dz.$$

Since $g_n(x-z)$ is a test function, we have, by (2.21),

$$a_n(x) = g_n(x).$$

Also

$$a_n(x) = \int_{-\infty}^{\infty} g_n(z)\,\delta(x-z)dz.$$

Hence

$$\delta(x) * g(x) = g(x) * \delta(x) = g(x), \text{ for all } g(x) \in \bar{S}.$$

Since $Z(\delta) = 1$, we have

$$Z(\delta(x) * g(x)) = Z(\delta)Z(g).$$

In a like manner we find that

$$\delta(x-x_o) * g(x) = g(x-x_o).$$

Since

$$Z(g(x-x_o)) = e^{-2\pi i x_o y} Z(g)$$

and also

$$Z(\delta(x-x_o)) = e^{-2\pi i x_o y},$$

we have

$$Z(\delta(x-x_o) * g(x)) = Z(\delta(x-x_o))Z(g(x)). \tag{2.25}$$

<u>Theorem 6</u>. Let a_j and x_j, $-M \leq j \leq N$, be constants and let

$$\Delta(x) = \sum_{j=-M}^{N} a_j \, \delta(x-x_j). \tag{2.26}$$

Then if $g(x)$ is any generalized function,

$$Z(\Delta(x) * g(x)) = Z(\Delta(x))Z(g(x)). \tag{2.27}$$

<u>Proof</u>: This follows easily from (2.25), and the linearity property of the convolution and of Z. For we obviously have

$$[a_1 \delta(x-x_1)+a_2 \delta(x-x_2)] * g(x) = a_1[\delta(x-x_1) * g(x)]+a_2[\delta(x-x_2) * g(x)].$$

By induction

$$\Delta(x) * g(x) = \sum_{j=-M}^{N} a_j[\delta(x-x_j) * g(x)].$$

Applying Z to both sides of the above equation and using its linearity property gives (2.27).

Let $H(y) = Z(\Delta(x))$ and $G(y) = Z(g(x))$. Noting that

$$\Delta(x) * g(x) = \sum_{j=-M}^{N} a_j g(x-x_j)$$

and applying Z^{-1} to both sides of (2.27), we have

$$\sum_{j=-M}^{N} a_j \quad g(x-x_j) = Z^{-1}(H(y)G(y)). \tag{2.28}$$

2.4 Trigonometric series.

If $f_z(x)$ is a generalized function for each value of the parameter z and if f(x) is a generalized function such that

$$\lim_{z\to a} \int_{-\infty}^{\infty} f_z(x)F(x)dx = \int_{-\infty}^{\infty} f(x)F(x)dx \tag{2.29}$$

for all $F(x) \in S$, the $f_z(x)$ is said to <u>converge</u> (weakly) to f(x) and we write

$$\lim_{z\to a} f_z(x) = f(x).$$

With this definition of convergence in $\overline{S}$, we have the following theorem.

<u>Theorem 7</u>. The trigonometric series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\pi\frac{x}{p}} \tag{2.30}$$

converges in the sense of (2.29) to a generalized function f(x) if and only if $a_n = O(|n|^N)$ for some N as $|n| \to \infty$. If (2.30) converges, then its Fourier transform is

$$g(y) = \sum_{n=-\infty}^{\infty} a_n \delta(y-\frac{n}{2p}). \qquad (2.31)$$

Also $f(x) = 0$ only if $a_n = 0$ for all n.

For a proof of this theorem, we refer the reader to [3], pp. 58-60.

The function $g(y)$ is called a "row of deltas" of spacing $\frac{1}{2p}$. This comes from the equality on any interval $(\frac{n}{2p},\frac{n+1}{2p})$ of $g(y)$ and an ordinary function which is zero on $(\frac{n}{2p},\frac{n+1}{2p})$. If $f(x)$ is a periodic and has a Fourier series representation, then the a_n are the Fourier coefficients

$$a_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{\frac{-in\pi x}{p}}dx.$$

This is equivalent to the statement that convergence in the ordinary (strong) sense implies convergence in the sense of (2.29) and that the limits are the same. The converse is not true, for by theorem 7

$\sum_{n=-\infty}^{\infty} \cos n\pi\frac{x}{p}$ converges in the sense of (2.29), but obviously not to an ordinary function.

2.5 The time-frequency interpretation of x and y.

If we take x to be time t, the function

$$g(t) = e^{2\pi i f_o t}$$

of frequency $f_o \geq 0$ has transform

$$G(y) = \delta(y-f_o).$$

In the sense of definition 8, $G(y) = 0$ for $y \neq f_o$, i.e., $G(y)$ displays

the frequency of $g(t)$. If

$$\bar{g}(t) = e^{-2\pi i f_o t},$$

then

$$\bar{G}(y) = \delta(y+f_o),$$

and $\bar{G}(y) = 0$ for $y \neq -f_o$. Hence, if we admit negative frequencies, $\bar{G}(y)$ displays the frequency of $\bar{g}(t)$, and we are led to interpret y as frequency.

If $h(t)$ is a function which can be written as

$$h(t) = \sum_{n=-M}^{N} a_n e^{2\pi i f_n t}$$

where M,N may be infinite provided the series converges in the sense of (2.29), then

$$H(y) = \sum_{n=-M}^{N} a_n \delta(y-f_n).$$

$H(y) = 0$ for $y \neq f_n$, $-M \leq n \leq N$. Thus $H(y)$ displays the frequencies f_n and the complex amplitudes a_n of the components of $h(t)$.

A Fourier series has for its transform a "row of deltas" (2.31), and thinking of the Fourier integral as the limit of a Fourier series as the period $2p$ tends to infinity (see [8] or [9]), the frequency interpretation of y carries over into integral sums. Using the symbol f instead of y, we have that if $h(t)$ is a function of time with Fourier transform $H(f)$, then $H(f)$ displays the frequencies and the complex amplitudes of the components of $h(t)$.

CHAPTER III

FILTERS

3.1 Linear systems.

A linear system, for our purposes, is a linear operator L which maps $\bar{S}$ into $\bar{S}$. By linear we mean that for all $f(t), g(t) \in \bar{S}$ and all scalars a,b

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]. \tag{3.1}$$

Let $L[\delta(t-\epsilon)] = h(t;\epsilon)$, and suppose that the family of functions $h(t;\epsilon)$ is known. Let $g(t)$ be an arbitrary function which we will refer to as the input to the linear system L, and let $f(t) = L[g(t)]$. $f(t)$ is called the output of L. Now since

$$g(t) = \int_{-\infty}^{\infty} g(\epsilon)\, \delta(t-\epsilon)\,d\epsilon,$$

we have

$$f(t) = L\left[\int_{-\infty}^{\infty} g(\epsilon)\, \delta(t-\epsilon)\,d\epsilon\right].$$

Assuming that (3.1) is sufficient to write

$$L\left[\int_{-\infty}^{\infty} g(\epsilon)\, \delta(t-\epsilon)\,d\epsilon\right] = \int_{-\infty}^{\infty} L[g(\epsilon)\, \delta(t-\epsilon)]\,d\epsilon,$$

then

$$f(t) = \int_{-\infty}^{\infty} g(\epsilon) L[\delta(t-\epsilon)]\,d\epsilon$$

$$= \int_{-\infty}^{\infty} g(\epsilon) h(t;\epsilon)\,d\epsilon.$$

Suppose that L satisfies the condition:

(A) If $L[g(t)] = f(t)$ and t_o is real constant, then $L[g(t-t_o)] = f(t-t_o)$, i.e., L is <u>time-invariant</u>.

Then if $L[\delta(t)] = h(t)$, $L\left\{\delta(t-t_o)\right\} = h(t-t_o)$ and

$$f(t) = \int_{-\infty}^{\infty} g(\epsilon)h(t-\epsilon)d\epsilon, \tag{3.2}$$

that is, the output of L is given in terms of the input and a unique function $h(t)$. The function $h(t)$ is called the <u>impulse response</u> or <u>weight function</u> of the linear system L, and its Fourier transform

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi ift}dt \tag{3.3}$$

is called the <u>system</u> or <u>transfer function</u> of L.

We note here the different uses of the symbol f. When used alone or as an argument, f denotes frequency. (See 3.3). When f is written with an argument, $f(t)$, it denotes a function of the time variable t. (See 3.2)

Note that (3.2) is the convolution $g*h$. If $f(t) \longleftrightarrow F(f)$ and $g(t) \longleftrightarrow G(f)$, then using (3.2) and assuming that the convolution theorem holds, we have

$$f(t) = g(t)*h(t) \longleftrightarrow G(f)H(f)$$

and

$$F(f) = G(f)H(f), \tag{3.4}$$

$$f(t) = \int_{-\infty}^{\infty} G(f)H(f)e^{2\pi ift}df.$$

That is, the Fourier transform of the output of the linear system L is equal to the product of the transforms of the input and the weight function $h(t)$. We also note that if $G(f)$ is the transform of an input

and $F(f)$ is the transform of a desired output, then from (3.4) the transfer function of the linear system L giving the desired output is

$$H(f) = \frac{F(f)}{G(f)} . \tag{3.5}$$

$H(f)$ may in general be complex [see (1.4)]

$$H(f) = A(f)\, e^{i\theta(f)}$$

where $A(f)$ and $\theta(f)$ have already been defined in the classical case as the Fourier spectrum and phase angle of $h(t)$, respectively.

Definition 9. A linear system L which satisfies (A) is called a filter if $A(f)$ is small in some sense on certain parts of the frequency axis. A low-pass filter is a filter for which $A(f)$ is small for $|f| > f_c$ where f_c is called the cut-off frequency. A band-pass filter is a filter for which $A(f)$ is small outside the intervals $[-f_c, -\overline{f}_c]$ and $[\overline{f}_c, f_c]$. A frequency $\overline{f}$ is said to be passed by a filter if $A(\overline{f})$ is not small.

3.2 Ideal low-pass filters.

We shall restrict our attention here to some particular cases where the phase angle $\theta(f)$ is constant,

$$\theta(f) = a. \tag{3.6}$$

Ideal smoothing filter.

This, by definition, is a low pass filter which passes all frequencies f such that $|f| \leq f_c$ without change and deletes all frequencies greater than f_c. No phase shift is involved, and hence $a = 0$.

Thus

$$H(f) = A(f) = \begin{cases} 1 & |f| \leq f_c \\ 0 & |f| > f_c . \end{cases} \tag{3.7}$$

See figure 3.1.

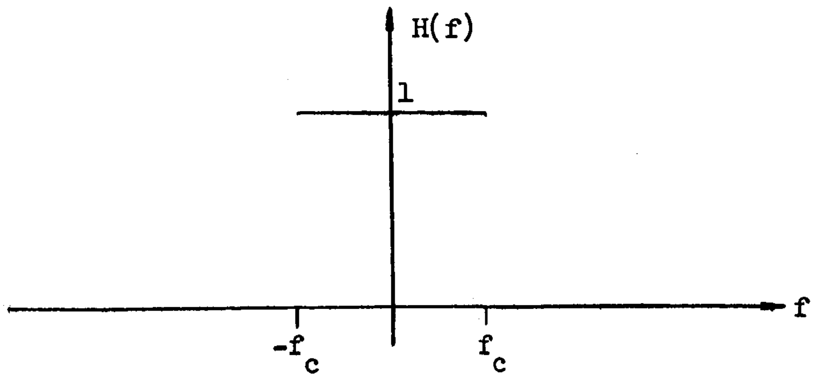


Figure 3.1.

The corresponding weight function is

$$h(t) = \int_{-f_c}^{f_c} e^{2\pi i f t} df \qquad (3.8)$$

$$= 2\int_0^{f_c} \cos 2\pi f t df$$

$$= \frac{\sin 2\pi f_c t}{\pi t},$$

or in terms of the angular frequency $w = 2\pi f$,

$$h(t) = \frac{\sin w_c t}{\pi t}. \qquad (3.9)$$

If $g(t)$ is the input to this filter, then the output is

$$f(t) = \int_{-\infty}^{\infty} g(z)h(t-z)dz,$$

which has transform [see (3.4)]

$$F(f) = \begin{cases} G(f)H(f) & |f| \leq f_c \\ 0 & |f| > f_c \end{cases}$$

where $g(t) \longleftrightarrow G(f)$. Hence $f(t)$ contains no frequencies greater than f_c.

Ideal smoothing and differentiating filter.

By theorem 3(b), if $g(t) \longleftrightarrow G(f)$, then

$$g'(t) \longleftrightarrow (2\pi i f)\, G(f).$$

Repeated application of this theorem yields

$$g^{(n)}(t) \longleftrightarrow (2\pi i f)^n G(f) \tag{3.10}$$

From (3.5) we see that to find the $n^{\underline{th}}$ derivative of an input $g(t)$ the transfer function must be $(2\pi i f)^n$. Then, in order to smooth using the ideal filter and find the $n^{\underline{th}}$ derivative, the transfer function is given by

$$H(f) = \begin{cases} (2\pi i f)^n & |f| \leq f_c \\ 0 & |f| > f_c \end{cases} \tag{3.11}$$

and the weight function is

$$h_n(t) = \int_{-f_c}^{f_c} (2\pi i f)^n \, e^{2\pi i f t} df. \tag{3.12}$$

But differentiating (3.8) n times, we have

$$h^{(n)}(t) = \int_{-f_c}^{f_c} (2\pi i f)^n \, e^{2\pi i f t} df \tag{3.13}$$

and so

$$h_n(t) = h^{(n)}(t) \tag{3.14}$$

Thus to find the weight function of the ideal smoothing and differentiating filter we simply differentiate the weight function of the smoothing filter the appropriate number of times.

Then

$$g^{(n)}(t) = \int_{-\infty}^{\infty} g(z)h^{(n)}(t-z)dz. \qquad (3.15)$$

3.3 The sampling theorem.

Ideal filters of the type discussed above are not physically realizable because of the jump discontinuities at $\pm f_c$. Futhermore, in digital filtering the input consists of a finite number of equally spaced values g_m, $M \le m \le N$, which we may assume are samples of some function $g(t)$ for $t = m\Delta t = \frac{m}{f_s}$. We may also assume that $g(t)$ defines a generalized function, for, recalling theorems 2 and 5, this does not place a serious restriction on $g(t)$. It is obvious that $g(t)$ is not uniquely determined by the values g_m, and hence the set of values g_m are associated with a subset G_{MN} of $\bar{S}$.

If we know that the samples g_m arise from a function $g(t)$ whose transform $G(f)$ is zero for $|f| > f_\alpha$, then the subset G_{MN} of $\bar{S}$ is reduced to a subset $G^1_{MN} \subset G_{MN}$. In this case $g(t)$ is said to be band-limited.

Theorem 8. Shannon's sampling theorem (see [5]).

If $g(t)$ is band-limited, i.e., if $g(t) \longleftrightarrow G(f)$ where

$$G(f) = 0 \qquad |f| \geqslant f_\alpha \qquad (3.16)$$

then $g(t)$ can be uniquely determined from its values

$$g_n = g\left(\frac{n}{2f_\alpha}\right) \qquad (3.17)$$

at a sequence of equidistant points of distance $\frac{1}{2f_\alpha}$ apart.

Futhermore

$$g(t) = \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi\ (2f_\alpha t-n)}{\pi\ (2f_\alpha t-n)} . \tag{3.18}$$

<u>Proof</u>: We first compute the g_n. We have, using (3.16),

$$g(t) = \int_{-f_\alpha}^{f_\alpha} G(f)e^{2\pi ift}df,$$

hence

$$g_n = g\left(\frac{n}{2f_\alpha}\right) = \int_{-f_\alpha}^{f_\alpha} G(f)e^{n\pi i\frac{f}{f_\alpha}} df. \tag{3.19}$$

Expanding G(f) in a Fourier series on $(-f_\alpha, f_\alpha)$ we have

$$G(f) = \sum_{n=-\infty}^{\infty} G_n e^{-n\pi i\frac{f}{f_\alpha}}, \quad -f_\alpha < f < f_\alpha, \tag{3.20}$$

where

$$G_n = \frac{1}{2f_\alpha} \int_{-f_\alpha}^{f_\alpha} G(f)e^{n\pi i\frac{f}{f_\alpha}} df \tag{3.21}$$

comparing (3.19) and (3.21), we have

$$G_n = \frac{g_n}{2f_\alpha} .$$

The function

$$\bar{G}(f) = \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha}\ e^{-n\pi i\frac{f}{f_\alpha}}, \quad -\infty < f < \infty,$$

is the periodic extension of G(f) and

$$\bar{G}(f) = G(f) \quad \text{for } -f_\alpha < f < f_\alpha.$$

Hence we may write

$$G(f) = H(f)\bar{G}(f)$$

where

$$H(f) = \begin{cases} 1 & |f| \le f_\alpha \\ 0 & |f| > f_\alpha . \end{cases}$$

Now [see (3.7) and (3.9)]

$$\frac{\sin 2\pi f_\alpha t}{\pi t} \longleftrightarrow H(f). \tag{3.24}$$

So we have

$$G(f) = H(f) \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha} e^{-n\pi i \frac{f}{f_\alpha}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha} H(f) e^{-n\pi i \frac{f}{f_\alpha}}$$

and

$$g(t) = \int_{-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha} H(f) e^{-n\pi i \frac{f}{f_\alpha}}] e^{2\pi i f t} df$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha} \int_{-\infty}^{\infty} H(f) e^{-n\pi i \frac{f}{f_\alpha}} e^{2\pi i f t} df.$$

Applying the t domain shifting theorem gives

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\alpha} \frac{\sin 2\pi f_\alpha (t - \frac{n}{2f_\alpha})}{\pi (t - \frac{n}{2f_\alpha})}$$

$$= \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi (2f_\alpha t - n)}{\pi 2f_\alpha t - n} .$$

If f_s is any number such that $f_s \geq 2f_\alpha$, then the theorem remains true if in the proof the periodic function $\overline{G}(f)$ is assumed to be of period f_s, and $H(f) = 1$ for $|f| \leq \frac{f_s}{2}$; $H(f) = 0$ for $|f| > \frac{f_s}{2}$. Therefore

$$g(t) = \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.25}$$

where

$$g_n = g(\frac{n}{f_s}). \tag{3.26}$$

If the g_n are known, as assumed above, for $M \leq n \leq N$, then the function

$$g_{MN}(t) = \sum_{n=M}^{N} g_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.27}$$

differs from each function $\overline{g}(t)$ of G^1_{MN} by

$$\epsilon_{\overline{g}}(t) = \sum_{n=-\infty}^{M-1} \overline{g}_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} + \sum_{n=N+1}^{\infty} \overline{g}_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.28}$$

where $\overline{g}_n = \overline{g}(\frac{n}{f_s})$. Hence, at least in the cases where the series in (3.25) converges uniformly to $g(t)$, the maximum difference

$$\operatorname*{Max}_{t} |\epsilon_{\overline{g}}(t)| = \max_{t} |g(t)-g_{MN}(t)| \tag{3.29}$$

can be made as small as we please by taking a sufficient number of terms in $g_{MN}(t)$. Hence we can associate with the samples $\{g_n\}$ a unique function $g(t)$ in the sense that (3.29) can be made arbitrarily small by taking a sufficient number of samples.

3.4 Definition of a digital filter.

Suppose that the sampled function $g(t)$ is band-limited. Then $G(f) = 0$ for $|f| > f_\alpha$. If $H(f)$ is a desired transfer function, then $H(f)G(f) = 0$ for $|f| > f_\alpha$. Thus if $\overline{H}(f)$ is a periodic extension of $H(f)$ with period $f_s \geq 2f_\alpha$, we have the transform $F(f)$ of the output $f(t)$ given by

$$F(f) = H(f)G(f) = \overline{H}(f)G(f), \tag{3.30}$$

for all f.

If $H(f)$ is such that $\overline{H}(f)$ can be written as a trigonometric series,

$$\overline{H}(f) = \sum_{n=-\infty}^{\infty} a_n e^{2n\pi i\frac{f}{f_s}} \tag{3.31}$$

with $a_n = O(|n|^N)$ for some N as $|n| \to \infty$, then, by theorem 7, $\overline{H}(f) \in \overline{S}$ and is the transform of

$$\overline{h}(t) = \sum_{n=-\infty}^{\infty} a_n \, \delta(t+\frac{n}{f_s}) \tag{3.32}$$

Now $g(t)$ is time sampled. In order to obtain a time sampled version of the output $f(t)$ we might try to define a convolution $\overline{h}(t)*g(t)$ and extend theorem 6 to functions $\Delta(t) = \overline{h}(t)$. Assuming that we could do this, we would have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n g(t+\frac{n}{f_s})$$

which would yield the sampled version of $f(t)$ for $t = \frac{m}{f_s}$ as

$$f(\frac{m}{f_s}) = \sum_{n=-\infty}^{\infty} a_n g(\frac{m+n}{f_s})$$

which is impossible to use digitally since it requires infinitely many samples.

Alternately, let

$$H_{MN}(f) = \sum_{n=M}^{N} a_n e^{2n\pi i \frac{f}{f_s}} \tag{3.33}$$

be a trigonometric polynomial which approximates H(f) in some sense. Then (3.33) is the transform of

$$h_{MN}(t) = \sum_{n=M}^{N} a_n \ \delta\left(t + \frac{n}{f_s}\right), \tag{3.34}$$

and we have shown that the convolution $h_{MN}(t) * g(t)$ is defined for all $g(t) \in \overline{S}$. Also, theorem 6 holds. Thus

$$F(f) = G(f)H(f) \doteq G(f)H_{MN}(f) = \overline{F}(f) \tag{3.35}$$

and $\overline{f}(t) \longleftrightarrow \overline{F}(f)$ is given by

$$\overline{f}(t) = h_{MN}(t) * g(t)$$

$$= \sum_{n=M}^{N} a_n \int_{-\infty}^{\infty} g(z) \ \delta\left(t - z + \frac{n}{f_s}\right) dz$$

$$= \sum_{n=M}^{N} a_n g\left(t + \frac{n}{f_s}\right).$$

For $t = \frac{m}{f_s}$, $\overline{f}_m = \overline{f}\left(\frac{m}{f_s}\right)$, $g_m = g\left(\frac{m}{f_s}\right)$, we have

$$\overline{f}_m = \sum_{n=M}^{N} a_n g_{m+n} \tag{3.36}$$

This is the fundamental formula of digital filtering.

Note that any pair (3.33) and (3.34) determine a linear operator L on $\bar{S}$ which satisfies condition (A) and, on the subspace G_s of all band-limited functions g(t) with $2f_\alpha \leq f_s$, acts as a low-pass filter. Now any finite set of constants a_n determines a function (3.34), which determines (3.33) and hence a linear operator L.

Definition 10. Let a_n, $M \leq n \leq N$, be any set of constants. Then the linear system $\bar{L}$ determined by the a_n is called a digital or numerical filter.

Application of $\bar{L}$ must be limited to the subspace G_s. Otherwise "frequency folding" occurs, i.e., frequencies in the intervals $\left(\frac{(2n-1)f_s}{2}, \frac{(2n+1)f_s}{2}\right)$ $n = \pm 1, \pm 2, \ldots$ are folded back into the $\left(-\frac{f_s}{2}, \frac{f_s}{2}\right)$. For example, suppose the input contains a frequency component $A\cos 2\pi(f_o+kf_s)t$ where $f_o < \frac{f_s}{2}$ and k is a positive integer. Then if we sample at $t = \frac{n}{f_s}$,

$$A\cos\left[2\pi(f_o+kf_s)\frac{n}{f_s}\right] = A\cos\left[2\pi f_o \frac{n}{f_s} + 2nk\pi\right]$$

$$= A\cos\left(2\pi f_o \frac{n}{f_s}\right).$$

The sample values would be the same as those obtained from a component $A\cos 2\pi f_o t$ for $t = \frac{n}{f_s}$. Hence the filter treats the frequency $f_o + kf_s > \frac{f_s}{2}$ in the same manner as f_o.

3.5 Even and odd transfer functions

In cases of interest here, the transfer function H(f) is either even or odd. Hence the trigonometric polynomial $H_{MN}(f)$ which approximates H(f) can be written in terms of $\cos 2n\pi\frac{f}{f_s}$ and $\sin 2n\pi\frac{f}{f_s}$

respectively. If we take M = -N some advantages are gained. Let

$$H_{MN}(f) = H_N(f) = \sum_{n=-N}^{N} a_n e^{2\pi ni\frac{f}{f_s}} . \tag{3.37}$$

For even functions,

$$H_N(f) = a_o + 2\sum_{n=1}^{N} a_n \cos 2n\pi\frac{f}{f_s}. \tag{3.38}$$

For odd functions,

$$H_N(f) = 2i\sum_{n=1}^{N} a_n \sin 2n\pi\frac{f}{f_s} . \tag{3.39}$$

Two questions now arise:

(1) given H(f), how are the coefficients a_n to be chosen, and

(2) what is the error introduced by the approximation $\overline{F}(f) \doteq F(f)$?

3.6 Methods of filter approximation

If H(f) is an ordinary function, there are currently two methods of approximating H(f) and obtaining the coefficients a_n. One of these methods--the Min-Max technique--is given by Martin [7]. Essentially, it assumes continuity of H(f) in which case, if $Q_n(f)$ is a set of N continuous and linearly independent functions on $[-\frac{f_s}{2}, \frac{f_s}{2}]$, then there exists a polynomial

$$P_N(f) = a_1Q_1(f) + \ldots + a_NQ_N(f)$$

which deviates the least from H(f) on $(-\frac{f_s}{2}, \frac{f_s}{2})$, i.e.,

$$\max_{f \in (-\frac{f_s}{2}, \frac{f_s}{2})} |H(f)-P_N(f)| \leq \max_{f \in (-\frac{f_s}{2}, \frac{f_s}{2})} |H(f) - \sum_{n=1}^{N} x_n Q_n(f)|$$

for any numbers $x_1, x_2, \ldots, x_N$. The $Q_n(f)$ are obtained after putting a constraint (or constraints) on a trigonometric polynomial (3.37). $P_N(f)$ is then fitted at a finite number of points to $H(f)$ in the above sense. A good approximation of the a_n is obtained by an iterative process, but the technique is long and complex, and not very versatile. That is, any change in $H(f)$ necessitates a complete repetition of the process for finding the a_n.

The alternate method assumes that $H(f)$ can be approximated by a Fourier series,

$$H(f) = \sum_{n=-\infty}^{\infty} h_n e^{2n\pi i \frac{f}{f_s}}, \tag{3.40}$$

where

$$h_n = \frac{1}{f_s} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} H(f) e^{-2n\pi i \frac{f}{f_s}} df, \tag{3.41}$$

and $H_N(f)$ is taken to be the truncated series for $H(f)$,

$$H_N(f) = \sum_{n=-N}^{N} h_n e^{2n\pi i \frac{f}{f_s}}. \tag{3.42}$$

This gives a function which is the best fit to $H(f)$ in the least mean square sense.

Noting that, since $H(f) = 0$ for $|f| > \frac{f_s}{2}$,

$$h(t) = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} H(f)e^{2n\pi i\frac{f}{f_s}}\, df, \tag{3.43}$$

and comparing (3.43) and (3.41), we see that

$$h_n = \frac{1}{f_s} h(-\frac{n}{f_s}). \tag{3.44}$$

This is the basic formula for computing the $h_n = a_n$ to use in (3.36). Therefore (3.36) can be written as

$$\bar{f}_m = \sum_{n=-N}^{N} h_n g_{m+n} . \tag{3.45}$$

3.7 Error analysis.

With an approximation $H_N(f)$ of $H(f)$, (3.35) becomes

$$F(f) = G(f)H(f) \doteq G(f)H_N(f) = \bar{F}(f),$$

and so

$$F(f)-\bar{F}(f) = G(f)[H(f)-H_N(f)].$$

This gives the error between the spectrum of the desired output and the spectrum of the actual output.

For a complex frequency component $g_o(t) = Ae^{2\pi i f_o t}$ in the input we have

$$g_o(t) = Ae^{2\pi i f_o t} \longleftrightarrow A\,\delta(f-f_o) = G(f_o)$$

and

$$F(f_o) = A\,\delta(f-f_o)H(f_o),$$

also

$$\bar{F}(f_o) = A\ \delta(f-f_o)H_N(f_o).$$

Denoting the difference in the outputs by $\epsilon(f_o,t)$ we have

$$|\ \epsilon(f_o,t)| = \left|\int_{-\infty}^{\infty} A\ \delta(f-f_o)[H(f_o)-H_N(f_o)]e^{2\pi ift}df\right|$$

$$= |Ae^{2\pi if_o t}[H(f_o) - H_N(f_o)]|$$

$$= |Ae^{2\pi if_o t}| \cdot |\ \epsilon(f_o,N)|,$$

where $\epsilon(f_o,N) = H(f_o) - H_N(f_o)$.

In the time sampled version:

$$|\ \epsilon(f_o, \frac{n}{f_s})| = |Ae^{2\pi if_o \frac{n}{f_s}}| \cdot |\ \epsilon(f_o,N)|. \qquad (3.46)$$

Thus the magnitude of the error in a component of the actual sampled output is given in terms of the magnitude of the corresponding component of the input function, and of the magnitude of the error in the approximation of $H(f)$.

Approximations of ϵ,

$$\epsilon = \max_f |\ \epsilon(f,N)| = \max_f |H(f) - H_N(f)| \qquad (3.47)$$

derived mathematically are usually found to be so large as to render them useless in applications. In applications of the filters discussed later, acceptable values of ϵ are in the range $.005 \leq \epsilon \leq .01$, or referred to unity, $\frac{1}{2}$%and 1%. When speaking of percent error we will always mean ϵ referred to unity. For a given $H(f)$, an N is

found empirically such that $H_N(f)$ approximates $H(f)$ within the desired limits.

However, satisfying the requirement that $.005 \leq \epsilon \leq .01$ does not imply the output error is within these bounds (see Chapter VII).

3.8 The Gibbs' phenomenon.

When approximating an ideal or designed transfer function $H(f)$ having one or more jump discontinuities with a truncated Fourier series, there exist oscillations in the approximating transfer function $H_N(f)$ near the discontinuities of $H(f)$ due to the Gibbs' phenomenon (see [10]). No matter how large N is taken, ϵ cannot be brought within the acceptable range $.005 \leq \epsilon \leq .01$. To avoid this difficulty $H(f)$ is first approximated by a function which is continuous. In most cases, this imposes a restriction on the input $g(t)$. The particular cases of interest here shall be dealt with in the next chapter.

CHAPTER IV

FILTER DESIGN

4.1 Assumptions about the input.

In order to apply a digital filter to a set of data $\{g_n\}$, we have made two assumptions about the data:

I. It arises from a function $g(t)$ which defines a generalized function, and

II. $g(t)$ is band-limited.

In many cases of interest, the Fourier spectrum $G(f)$ of a signal $g(t)$ consists of a desired signal spectrum in an interval $[-f_c,f_c]$, an unwanted signal spectrum (noise spectrum) in intervals $[-f_\alpha,f_c)$ and $(f_c,f_\alpha]$, and $G(f) = 0$ for $|f| > f_\alpha$. When applying a low-pass filter, elimination of the unwanted spectrum is desired. Hence the ideal filter transfer function, $H_I(f)$, is such that $H_I(f) = 0$, $|f| > f_c$. Usually $H_I(\pm f_c) \neq 0$ and $H_I(f)$ has jump discontinuities at $f = \pm f_c$. If the truncated Fourier series of $H_I(f)$ is used to approximate $H_I(f)$, then, due to the Gibbs' phenomenon, large oscillations persist in a neighborhood of $\pm f_c$. Futhermore, the amplitude of these oscillations remains constant with increasing N. The truncated Fourier series is continuous everywhere because it is a finite sum of everywhere continuous functions. Since $H_I(f_c) \neq 0$, we expect that the truncated series, $H_N(f)$, is such that $H_N(f_c) \neq 0$. Then, by continuity, $H_N(f)$ is non-zero on some interval $(f_c,f_c+\Delta f)$ where $\Delta f > 0$ and depends on N. Any unwanted frequencies which appear in this interval are passed--though somewhat attenuated--by

the approximating filter. Hence, in addition to the large oscillations which appear near $\pm f_c$, unwanted frequencies arbitrarily close to $\pm f_c$ cannot be eliminated by increasing N. This undesirable property must be tolerated because it is a property of any truncated Fourier series such that $H_N(f_c) \neq 0$. However, the large oscillations are caused by non-uniform convergence of the Fourier series of $H_I(f)$. This can be remedied by redefining $H_I(f)$ so that it is a continuous function. We choose to do this on the intervals $[-f_c-\Delta f, -f_c)$ and $(f_c, f_c+\Delta f)$ for some $\Delta f > 0$. Any unwanted frequencies in these intervals will be passed to some extent by the filter, but, as pointed out above, this cannot be avoided anyway. However, in many applications unwanted frequencies do not appear near $\pm f_c$. Therefore, we make the following third assumption about the data:

III. The desired signal spectrum and the unwanted spectrum of $g(t)$ are disjoint.

Then there exists a $\Delta f > 0$ such that the signal spectrum $G(f) = 0$ on $(-f_c-\Delta f, -f_c)$ and $(f_c, f_c+\Delta f)$. Letting $f_T = f_c+\Delta f$, we may modify $H_I(f)$ on $[-f_T, -f_c)$ and $(f_c, f_T]$ to obtain a function $H(f)$ continuous for all f and thereby eliminate the Gibbs' phenomenon. $H(f)$, as defined on the intervals $[-f_T, -f_c)$ and $(f_c, f_T]$, is called the <u>roll-off</u> of the filter, and the frequency f_T is called the <u>termination frequency</u>.

4.2 <u>Filter design by convolution</u>.

The usual approach to the design of a filter is to select the ideal transfer function $H_I(f)$ on $[-f_c, f_c]$ and then to specify the roll-off. This gives the filter transfer function $H(f)$ from which the weight function $h(t)$ is found. The weights of the filter to be used in (3.45) are then computed from (3.44). In addition to not being very versatile, this approach usually involves some rather long and tedious integration in determining $h(t)$.

We propose a different approach to the design which simplifies the integration and gives considerable freedom in varying the roll-off shape of the filter. We shall use the convolution theorem of Chapter I:

$$g(t)k(t) \longleftrightarrow \int_{-\infty}^{\infty} G(f-z)K(z)\,dz \tag{4.1}$$

where $g(t) \longleftrightarrow G(f)$ and $k(t) \longleftrightarrow K(f)$.

We note here that filters for simultaneously performing smoothing and differentiation can be found from the weight-transfer functions, $h(t)$ and $H(f)$, of the smoothing filter in a manner analogous to that in the ideal case [see section 3.2]. That is, to smooth and find the $n^{\underline{th}}$ derivative, the transfer function is

$$Y^n(f) = (2\pi i f)^n H(f). \tag{4.2}$$

With

$$y^n(t) \longleftrightarrow Y^n(f),$$

we have

$$y^n(t) = h^{(n)}(t) \tag{4.3a}$$

where

$$h(t) \longleftrightarrow H(f).$$

In (4.3a), let $t = \frac{-x}{f_s}$. Then $t^n = \left(\frac{-x}{f_s}\right)^n$ and $dt^n = \left(-\frac{1}{f_s}\right)^n dx^n$.

Hence

$$y^n\left(\frac{-x}{f_s}\right) = \frac{d^n h\left(\frac{-x}{f_s}\right)}{\left(-\frac{1}{f_s}\right)^n dx^n}$$

$$= (-1)^n f_s{}^n \frac{d^n h\left(\frac{-x}{f_s}\right)}{dx^n}$$

Using (3.44) to compute the weights of the filter, we have

$$y_k^n = \frac{1}{f_s} y^n\left(\frac{-k}{f_s}\right)$$

$$= \frac{1}{f_s} y^n\left(\frac{-x}{f_s}\right)\Big]_{x=k}$$

$$= (-1)^n f_s{}^n \frac{\left\{d^n \frac{1}{f_s} h\left(\frac{-x}{f_s}\right)\right\}}{dx^n}\Big]_{x=k}.$$

We now see that we may write

$$y_k^n = (-1)^n f_s{}^n \frac{d^n h_k}{dk^n} \tag{4.3b}$$

where $h_k = \frac{1}{f_s} h\left(\frac{-k}{f_s}\right)$ and, for purposes of differentiating, k is treated as a variable in the right side of (4.3b).

Returning to the problem of designing the filter, we conclude from the above that we may restrict ourselves to the design of smoothing filters. Hence suppose that

$$H(f) = \int_{-\infty}^{\infty} G(f-z)K(z)dz. \tag{4.4}$$

Ideally, for smoothing we want H(f) to be continuous, and

$$H(f) = \begin{cases} 1, & 0 \leq f \leq f_c, \\ \text{monotonic decreasing}, & f_c < f < f_T, \\ 0, & f \geq f_c, \\ H(-f), & f < 0. \end{cases} \tag{4.5}$$

We attempt to find function G(f) and K(f) such that H(f) given by (4.4) has these properties.

Then the weight function h(t) is given by

$$h(t) = g(t)k(t). \tag{4.6}$$

Let

$$G(f) = \begin{cases} 1, & |f| \le \frac{f_c+f_T}{2}, \\ 0 & |f| > \frac{f_c+f_T}{2}. \end{cases} \tag{4.7}$$

Then comparing with (3.7) and (3.8), we see that

$$g(t) = \frac{\sin \pi(f_T+f_c)t}{\pi t}.$$

Noting that $G(f-z) = 0$ for $|f-z| > \frac{f_T+f_c}{2}$, (4.4) becomes

$$H(f) = \int_{f-\frac{f_T+f_c}{2}}^{f+\frac{f_T+f_c}{2}} K(z)dz. \tag{4.8}$$

To find $H(f_o)$, $K(z)$ is integrated over an interval of length (f_T+f_c) with f_o as its mid-point. Any function $K(z)$ which is zero for $|z| > \frac{f_T-f_c}{2} = \frac{\Delta f}{2}$ and is an even function of z with area 1 on $[\frac{-\Delta f}{2},\frac{\Delta f}{2}]$ yields a satisfactory H(f).

Filter 1. The Ormsby smoothing filter (p=1).

In (4.8) let

$$K(f) = K_1(f) = \begin{cases} \frac{1}{\Delta f}, & |f| \leq \frac{\Delta f}{2}, \\ 0, & |f| > \frac{\Delta f}{2} \end{cases} \tag{4.9}$$

See figure 4.1.

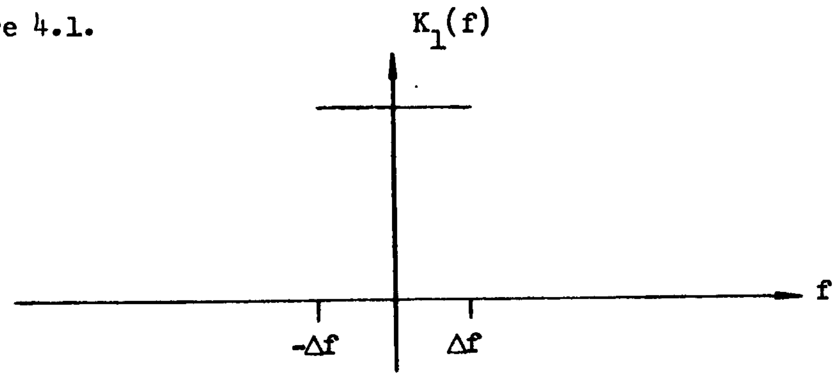


Figure 4.1.

Then

$$k_1(t) = \int_{\frac{-\Delta f}{2}}^{\frac{\Delta f}{2}} \frac{1}{\Delta f} e^{2\pi i f t} df$$

$$\frac{2}{\Delta f} \int_0^{\frac{\Delta f}{2}} \cos (2\pi f t) df$$

$$= \frac{1}{\pi \Delta f t} \left[\sin 2\pi f t \right]_0^{\frac{\Delta f}{2}}$$

$$= \frac{\sin \pi \Delta f t}{\pi \Delta f t} ,$$

and

$$h_1(t) = k_1(t)g(t)$$

$$= \frac{\sin \pi\Delta ft \sin \pi(f_T+f_c)t}{\pi^2\Delta ft^2}. \tag{4.10}$$

Changing to the angular frequency $w = 2\pi f$, $\Delta w = 2\pi\Delta f$, $w_T = 2\pi f_T$, $w_c = 2\pi f_c$, we have

$$h_1(t) = \frac{2 \sin \frac{\Delta wt}{2} \sin \frac{(w_T+w_c)}{2}}{\pi\Delta wt^2},$$

and applying a well-known trigonometric identity

$$h_1(t) = \frac{\cos w_c t - \cos w_T t}{\pi\Delta wt^2}. \tag{4.11}$$

This is the weight function given by Ormsby [11] for p=1. The corresponding transfer function as a function of f is

$$H_1(f) = \begin{cases} 1, & |f| \leq f_c, \\ 0, & |f| > f_T, \\ \frac{f+f_T}{\Delta f}, & -f_T \leq f < -f_c, \\ \frac{f_T-f}{\Delta f}, & f_c < f \leq f_T. \end{cases}$$

$H_1(f)$ has a straight line roll-off (see figure 4.2).

Replacing f by $2\pi f = w$ gives the transfer function as given by Ormsby in terms of w. We remind the reader that, when using the angular

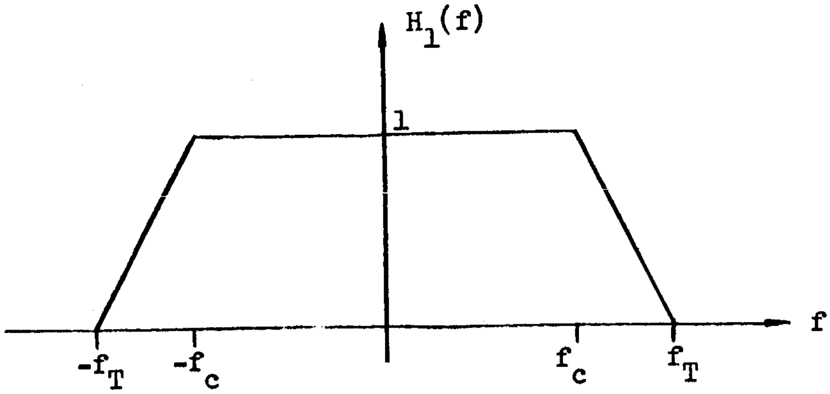


Figure 4.2.

frequency $w = 2\pi f$, a factor of $\frac{1}{2\pi}$ appears in the statement of the Fourier integral theorem, i.e., if

$$\bar{G}(w) = \int_{-\infty}^{\infty} g(t)e^{-iwt}dt,$$

then

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(w)e^{iwt}dw.$$

In the statement of the theorem in Chapter I, we have

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}dt.$$

Comparing, we see that

$$G(f) = \bar{G}(w) = \bar{G}(2\pi f).$$

For convenience, we shall use the same symbol G for both G and $\bar{G}$. Then the argument of G determines which form of the theorem to use, i.e., G(f) indicates the form of Chapter I is to be used and G(w) indicates that the above form is to be used.

Finally, note that $\frac{dH_1(f)}{df}$ is discontinuous at $\pm f_c$ and $\pm f_T$.

Filter 2. The Martin-Graham smoothing filter.

In (4.8) let

$$K(f) = K_2(f) = \begin{cases} \frac{\pi}{2\Delta f} \cos \frac{\pi f}{\Delta f}, & |f| \le \frac{\Delta f}{2}, \\ 0, & |f| > \frac{\Delta f}{2}. \end{cases} \tag{4.12}$$

See figure 4.3.

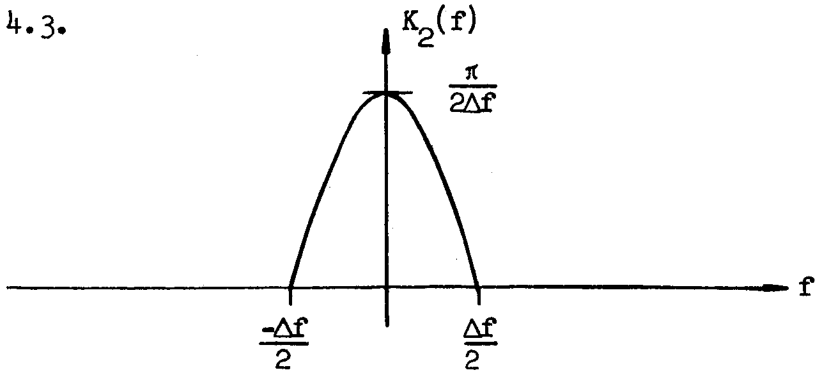


Figure 4.3.

Then

$$k_2(t) = \int_{\frac{-\Delta f}{2}}^{\frac{\Delta f}{2}} \frac{\pi}{2\Delta f} \cos \frac{\pi f}{\Delta f}\, e^{2\pi i f t} df$$

$$= \frac{\pi}{\Delta f} \int_0^{\frac{\Delta f}{2}} \cos \frac{\pi f}{\Delta f} \cos 2\pi f t \, df$$

$$= \frac{\pi}{\Delta f} \left[\frac{\sin\left(\frac{\pi}{\Delta f} - 2\pi t\right) f}{2\left(\frac{\pi}{\Delta f} - 2\pi t\right)} + \frac{\sin\left(\frac{\pi}{\Delta f} + 2\pi t\right) f}{2\left(\frac{\pi}{\Delta f} + 2\pi t\right)} \right]_0^{\frac{\Delta f}{2}}$$

$$= \frac{1}{2\Delta f}\left[\frac{\sin(\frac{\pi}{2} - \pi\Delta ft)}{(\frac{1}{\Delta f} - 2t)} + \frac{\sin(\frac{\pi}{2} + \pi\Delta ft)}{(\frac{1}{\Delta f} + 2t)}\right]$$

$$= \frac{1}{2\Delta f}\left[\frac{\cos \pi\Delta ft}{(\frac{1}{\Delta f} - 2t)} + \frac{\cos \pi\Delta ft}{(\frac{1}{\Delta f} + 2t)}\right]$$

$$= \frac{\cos \pi\Delta ft}{(1-4\Delta f^2t^2)} \; .$$

Then

$$h_2(t) = k_2(t)g(t)$$

$$= \frac{\cos \pi\Delta ft \sin \pi(f_T+f_c)t}{\pi t(1-4\Delta f^2t^2)} \, , \qquad (4.13)$$

where $\Delta f^2 = (\Delta f)^2$. We will also use the notation $\Delta w^2 = (\Delta w)^2$. Letting $w = 2\pi f$ gives

$$h_2(t) = \frac{\cos \frac{\Delta wt}{2} \sin \frac{(w_T+w_c)t}{2}}{\pi t(1- \frac{\Delta w^2t^2}{\pi^2})}$$

and using a well-known trigonometric identity gives, after simplifying,

$$h_2(t) = \frac{\pi(\sin w_c t + \sin w_T t)}{2t(\pi^2 - \Delta w^2t^2)}. \qquad (4.14)$$

This is the form of the weight function given by Graham [12].

The form given by Martin [6], [7] is obtained from (4.13) by going to the frequency ratio $r = \frac{f}{f_s}$, $2h = r_d = \frac{\Delta f}{f_s}$, etc., and computing

$$h_n = \frac{1}{f_s} h_2(-\frac{n}{f_s})$$

$$= \frac{1}{f_s} \frac{\cos[\pi(\Delta r f_s)(-\frac{n}{f_s})] \sin[\pi f_s(r_c+r_T)(-\frac{h}{f_s})]}{\pi(-\frac{n}{f_s})(1-4r_d{}^2 f_s{}^2 \frac{n^2}{f_s{}^2})}$$

$$= \frac{\cos[n\pi r_d] \sin[n\pi(2r_c+r_d)]}{n\pi(1-4r_d{}^2 n^2)} \, . \qquad (4.15)$$

This is a convenient expression for computing the weights h_n of the filter. The value of h_o is computed by using L'Hospital's rule, and

$$h_o = 2r_c+r_d$$

$$= \frac{w_T+w_c}{2\pi}. \qquad (4.16)$$

The same procedure must be used for finding h_m if $m = \frac{1}{2r_d}$, giving

$$h_m = 2r_d{}^2 \sin(\frac{\pi(2r_c+r_d)}{2r_d}). \qquad (4.17)$$

The transfer function of this filter, in terms of w, is

$$H_2(w) = \begin{cases} 1 & |w| \leq w_c, \\ 0; & |w| > w_T, \\ \frac{1}{2}[1+\cos\frac{\pi(w-w_c)}{\Delta w}], & w_c < w < w_T, \\ \frac{1}{2}[1+\cos\frac{\pi(w+w_c)}{\Delta w}], & -w_T < w < -w_c \, . \end{cases} \qquad (4.18)$$

See figure 4.4.

Alternate expressions for the roll-off are

$$\frac{1}{2}[1+\cos\frac{\pi(w+w_c)}{\Delta w}] = \cos^2(\frac{\pi(w-w_c)}{2\Delta w}).$$

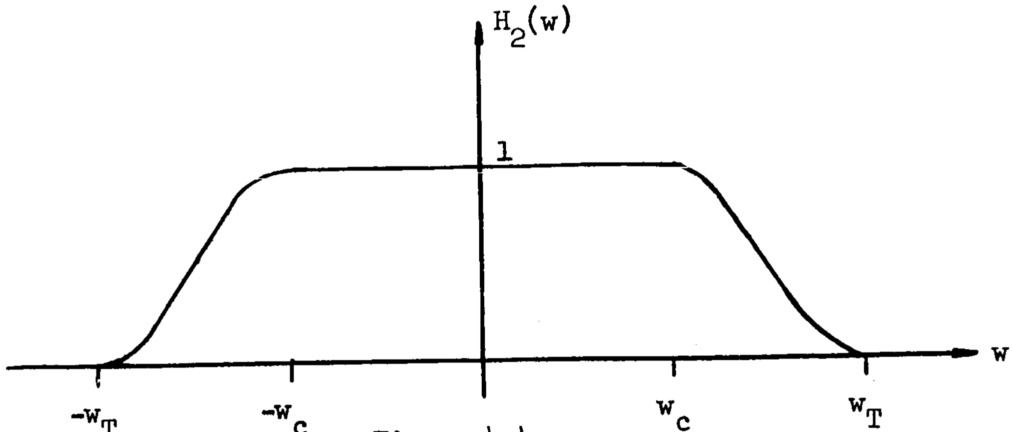


Figure 4.4.

and

$$\frac{1}{2}\left[1+\cos\frac{\pi(w+w_c)}{\Delta w}\right] = \cos^2\left(\frac{\pi(w+w_c)}{2\Delta w}\right).$$

Note that $H_2(w)$ has one continuous derivative, and $\frac{d^2H_2(w)}{dw^2}$ is discontinuous at $\pm\, w_T$ and $\pm\, w_c$.

4.3 Comparison of the performance of the Ormsby and Martin-Graham smoothing filters.

A comparison of the above filters can be drawn by expressing $H_{1,N}(f)$ and $H_{2,N}(f)$, the truncated Fourier series for $H_1(f)$ and $H_2(f)$, respectively, in integral form. We expand $K_1(f)$ and $K_2(f)$ in a Fourier series, then truncating these series gives:

$$H_{1,N}(f) = \int_{f-\frac{f_T+f_c}{2}}^{f+\frac{f_T+f_c}{2}} K_{1,N}(z)\,dz, \tag{4.19}$$

$$H_{2,N}(f) = \int_{f-\frac{f_T+f_c}{2}}^{f+\frac{f_T+f_c}{2}} K_{2,N}(z)\,dz \tag{4.20}$$

Since $K_{1,N}(z)$ is the truncated series of a function with jump discontinuities at $\pm \frac{\Delta f}{2}$ [see(4.9)], the Gibbs' phenomenon is present. Hence overshoot is present near $\pm \frac{\Delta f}{2}$, the amplitude of which can not be reduced by increasing N. We can expect some relatively large oscillations to be present, at least for small values of N, in $H_{1,N}(f)$. $K_2(z)$ is continuous, and the amplitude of the oscillations of $K_{2,N}(z)$ decreases monotonically with increasing N. Hence we expect the Martin-Graham filter to perform better than the Ormsby (p=1) filter. The results of comparative programs where the truncated series (4.19) and (4.20) were computed at equidistant points indicate that this conclusion is true. For $\epsilon = .01$, over 50% more weights were required by the Ormsby filter.

4.4 Some new smoothing filters

We shall give, without performing the details of integration, several new designs which are of some interest. The transfer function-weight function pair will be given in terms of the angular frequency $w = 2\pi f$.

Filter 3. Let

$$K_3(w) = \begin{cases} \frac{4\pi}{\Delta w} \cos^2(\frac{\pi w}{\Delta w}) & |w| \leq \frac{\Delta w}{2} \\ 0 & |w| > 0 \end{cases} . \tag{4.21}$$

Then

$$k_3(t) = \frac{4\pi^2}{4\pi^2 - \Delta w^2 t^2} \cdot \frac{2 \sin \frac{\Delta w}{2} t}{\Delta w t}$$

$$h_3(t) = k_3(t) g(t)$$

$$= \left(\frac{4\pi^2}{4\pi^2-\Delta w^2t^2}\right)\left(\frac{2\sin\frac{\Delta wt}{2}\sin\frac{w_T+w_c}{2}t}{\Delta w\pi t^2}\right)$$

$$= \frac{4\pi^2}{4\pi^2-\Delta w^2t^2}\cdot h_1(t) \tag{4.22}$$

where $h_1(t)$ is the Ormsby weight function (4.11). The roll-off of $H_3(w)$ is given by

$$\frac{1}{2\pi}\sin\frac{2(w-w_c)}{\Delta w}\pi + \frac{w_T-w}{\Delta w}, \qquad w_c < w \leq w_T,$$

and $H_3(w)$ has two continuous derivatives. (see figure 4.5)

<u>Filter 4.</u> Let

$$K_4(w) = \begin{cases} \frac{3\pi^2}{2\Delta w}\cos^3\left(\frac{\pi w}{\Delta w}\right), & |w| \leq \frac{\Delta w}{2}, \\ 0 & |w| > \frac{\Delta w}{2}. \end{cases} \tag{4.23}$$

Then

$$k_4(t) = \frac{9\pi^2}{9\pi^2-\Delta w^2t^2}\cdot\frac{\pi^2\cos\frac{\Delta wt}{2}}{(\pi^2-\Delta w^2t^2)},$$

and

$$h_4(t) = k_4(t)g(t)$$

$$= \frac{9\pi^2}{9\pi^2-\Delta w^2t^2}\left[\frac{\pi^2\cos\frac{\Delta wt}{2}}{\pi^2-\Delta w^2t^2}\cdot\frac{\sin\left(\frac{w_T+w_c}{2}\right)t}{\pi t}\right]$$

$$= \frac{9\pi^2}{9\pi^2-\Delta w^2t^2}\ h_2(t), \tag{4.24}$$

where $h_2(t)$ is the Martin-Graham weight function (4.14). The transfer function $H_4(w)$ has three continuous derivatives and the roll-off is given by:

$$H_4(w) = \frac{9}{16} \cos \left(\frac{(w-w_c)}{\Delta w}\pi\right) - \frac{1}{16} \cos \left(\frac{3(w-w_c)}{\Delta w}\pi \right) + \frac{1}{2}$$

for $w_c < w \leq w_T$. This is shown in figure 4.5.

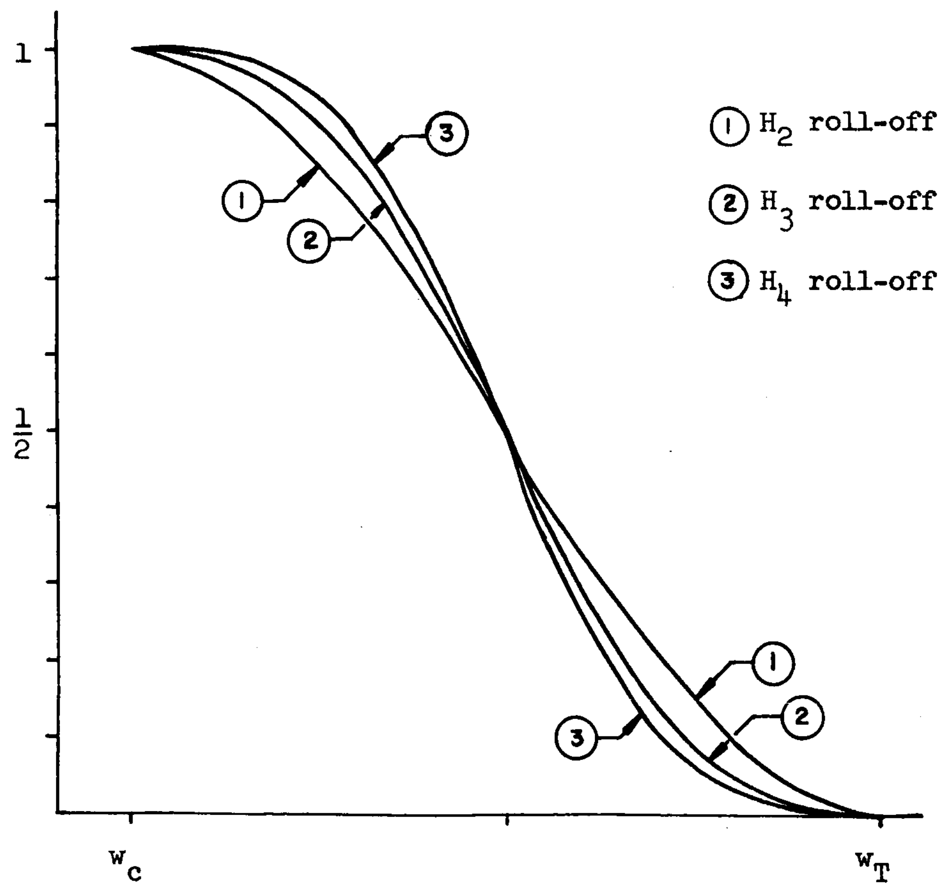


Figure 4.5

Filter 5. Let

$$K_5(w) = \begin{cases} \frac{3\pi}{\Delta w}\left(1 - \frac{4w^2}{\Delta w^2}\right), & |w| \leq \frac{\Delta w}{2}, \\ 0, & |w| > \frac{\Delta w}{2}. \end{cases} \tag{4.25}$$

This gives a weight function, where $\Delta w^3 = (\Delta w)^3$,

$$h_5(t) = \frac{12}{\pi \Delta w^3 t^4}\left[\sin\left(\frac{w_T + w_c}{2}\right)t\right]\cdot\left[2 \sin\frac{\Delta w}{2} t - t\Delta w \cos\frac{\Delta w}{2} t\right]. \tag{4.26}$$

The roll-off of $H_5(w)$, $w_c < w \leq w_T$, is a third degree polynomial and is essentially the same as that of $H_2(w)$.

Using the quantity ϵ defined by (3.46) as a measure of the performance of a filter to compare the above filters, one is led to the following conclusions:

1) The Martin-Graham filter gives $\epsilon = .01$ with smaller N than any of the others. In fact, out of numerous designs none has been found which gives $\epsilon = .01$ for smaller N than this filter. The performance of filter 5 is essentially the same, the ϵ values differing slightly in the third decimal place.

2) Filters 3 and 4 give values of $\epsilon \leq .005$ for smaller N than the Martin-Graham filter and filter 5.

3) In no case did the Ormsby filter perform as well as the other filters.

In comparison with the Martin-Graham filter, the only advantage filter 5 has is that no special evaluation for h_n, $n \neq 0$, is required; h_o is the same for all the above filters. In addition to the improved performance for $\epsilon \leq .005$, useable error bounds can be found for filters 3 and 4 without resorting to empirical methods.

4.5 Some smoothing error bounds

Except for filter 5, each of the above weight functions are of the form

$$h(t) = \frac{k(t)}{P(t)} ,$$

where $k(t)$ is an expression containing sums and products of trigonometric functions of t and $P(t)$ is a polynomial in t. The Fourier coefficients of $H(f)$ computed from $h(t)$ retain this character,

$$h_n = \frac{1}{f_s} h(\frac{-n}{f_s}) = \frac{1}{f_s} \frac{k(\frac{-n}{f_s})}{P(\frac{-n}{f_s})} .$$

Now the error as a function of f and N is

$$\epsilon(f,N) = H(f) - H_N(f)$$

$$= 2 \sum_{n=N+1}^{\infty} h_n \cos 2n\pi \frac{f}{f_s}$$

$$= \frac{2}{f_s} \sum_{n=N+1}^{\infty} \frac{k(-\frac{n}{f_s})}{P(-\frac{n}{f_s})} \cos 2n\pi\frac{f}{f_s} .$$

Letting $A = \max\limits_{n,f} |k(-\frac{n}{f_s}) \cos 2n\pi \frac{f}{f_s} |$, we have

$$\epsilon = \max_f |\epsilon(f,N)| \leq \frac{2A}{f_s} \sum_{n=N+1}^{\infty} |\frac{1}{P(\frac{-n}{f_s})}| . \qquad (4.27)$$

If $|P(t)| > 0$ for $t > \frac{N}{f_s}$, the sum in (4.24) can be approximated by

$$\left| \int_N^{\infty} \frac{dx}{P(\frac{-x}{f_s})} \right| .$$

Martin-Graham bound

The above method gives (see figure 4.6)

$$\epsilon \leq \frac{1}{\pi} \log \frac{4N^2 \Delta f^2}{4N^2 \Delta f^2 - f_s^{\ 2}} \tag{4.28}$$

For $\epsilon = .01$, the predicted value of N is

$$N \geq \frac{2.85\ f_s}{\Delta f} \tag{4.29}$$

and for $\epsilon = .005$

$$N \geq \frac{4f_s}{\Delta f} \tag{4.30}$$

These values of N are much too large. It has been determined empirically that $N \geq \dfrac{1.25\ f_s}{\Delta f}$ gives $.005 < \epsilon < .01$.

Filter 3.

For this filter,

$$\epsilon \leq \frac{1}{\pi^2} \left\{ \log \left[\frac{N\Delta f + f_s}{N\Delta f - f_s} \right] - \frac{2f_s}{N\Delta f} \right\} . \tag{4.31}$$

For $\epsilon = .01$, the predicted value of N is

$$N \geq \frac{2f_s}{\Delta f}, \tag{4.32}$$

and for $N \geq \dfrac{3f_s}{\Delta f}$, $\epsilon < .003$ (see figure 4.6).

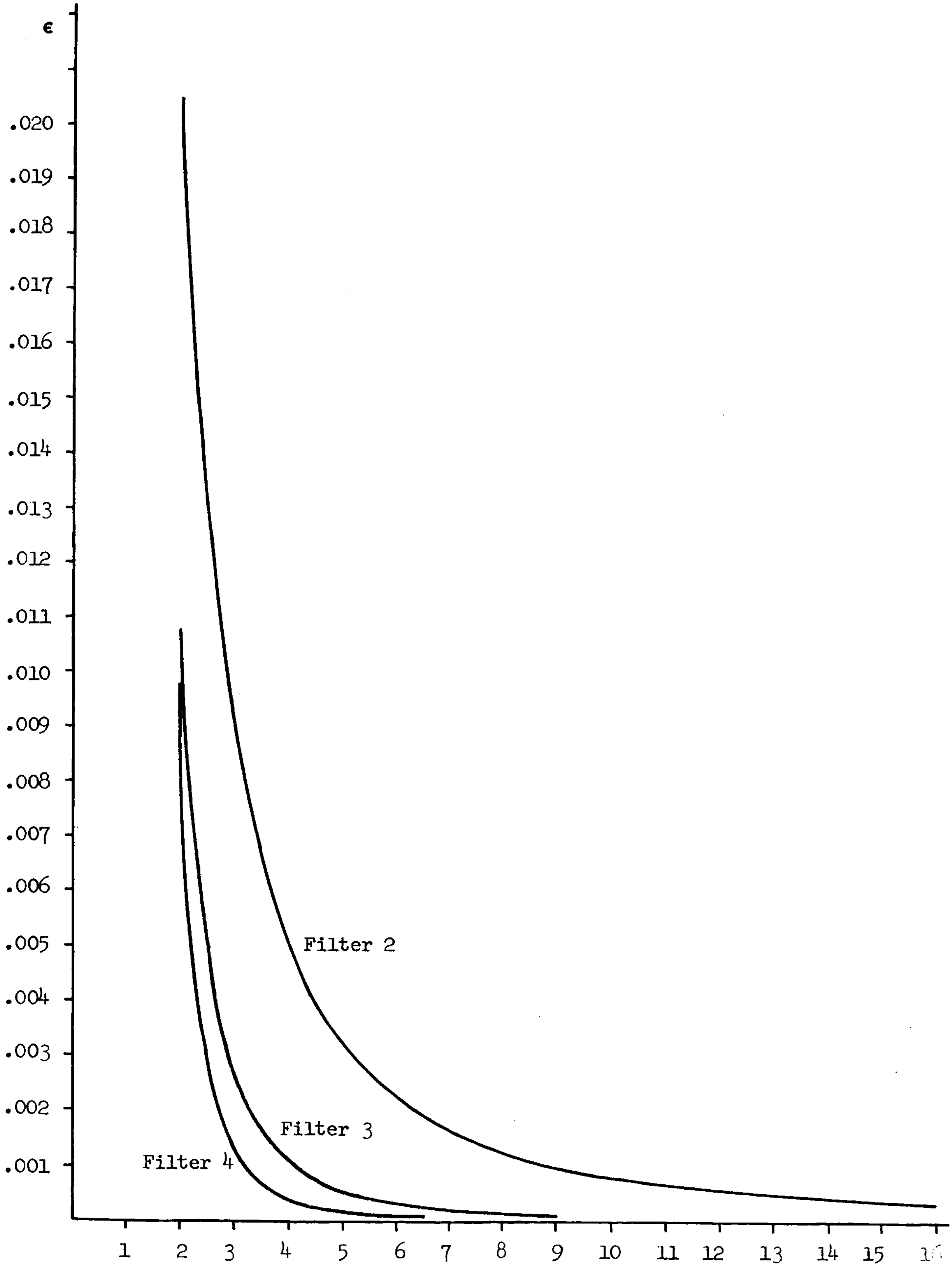


Figure 4.6

$N\Delta f/f_s$

Filter 4.

For this filter

$$\epsilon \leq \frac{1}{8\pi} \Big\{ 9 \log [4\pi^2 N^2 \Delta f^2 - \pi^2 f_s^{\,2}] - 16 \log [2\pi N \Delta f]$$

$$- \log [4\pi^2 N^2 \Delta f^2 - 9\pi^2 f_s^{\,2}] \Big\}. \tag{4.33}$$

For $\epsilon = .01$, the predicted value of N is the same as in (4.32).

For $N \geq \frac{3f_s}{\Delta f}$, (4.33) gives $\epsilon < .0014$ (see figure 4.6).

4.6 Smoothing filter constraints

In general, a signal $g(t)$ may have a polynomial content, and in such cases $g(t)$ is not band-limited. Denoting the polynomial content of $g(t)$ by $P(t)$, if

$$g(t) = \bar{g}(t) + P(t) \tag{4.34}$$

where $\bar{g}(t)$ is a band-limited function, then the weights can be constrained so that the sampled values $P(m\Delta t), \Delta t = \frac{1}{f_s}$, are passed without error.

We recall that the output of a digital filter is given by

$$f_m = \sum_{n=-N}^{N} h_n g_{m+n}.$$

Applying this to the sampled version of (4.34) gives

$$f_m = \sum_{n=-N}^{N} h_n [\bar{g}_{m+n} + P[(m+n)\Delta t]]$$

$$= \sum_{n=-N}^{N} h_n \bar{g}_{m+n} + \sum_{n=-N}^{N} h_n P[(m+n)\Delta t]. \tag{4.35}$$

Since $\overline{g}(t)$ is band-limited, the first term on the right side of (4.35) poses no problems. We want the second term to be $P(m\Delta t)$. Assuming that $P(t)$ is of degree p,

$$P(t) = \sum_{j=0}^{p} a_j t^j. \tag{4.36}$$

We want

$$P(m\Delta t) = \sum_{j=0}^{p} a_j (m\Delta t)^j$$

$$= \sum_{n=-N}^{N} h_n \sum_{j=0}^{p} a_j (m+n)^j \Delta t^j.$$

Interchanging the summation gives

$$\sum_{j=0}^{p} a_j (m\Delta t)^j = \sum_{j=0}^{p} a_j \sum_{n=-N}^{N} h_n (m+n)^j \Delta t^j. \tag{4.37}$$

We see from (4.37) that it suffices to consider the $k^{\underline{th}}$ term

$$m^k \Delta t^k = \sum_{n=-N}^{N} h_n (m+n)^k \Delta t^k$$

or, dividing by Δt^k,

$$m^k = \sum_{n=-N}^{N} h_n (m+n)^k.$$

Expanding $(m+n)^k$ and summing each term gives

$$m^k = m^k \sum_{n=-N}^{N} h_n + km^{k-1} \sum_{n=-N}^{N} nh_n + \ldots$$

$$+\binom{k}{r} m^{k-r} \sum_{n=-N}^{N} n^r h_n + \ldots \sum_{n=-N}^{N} n^k h_n. \qquad (4.38)$$

From (4.38) we see that it suffices to have

A:

$$\sum_{n=-N}^{N} h_n = 1 \qquad (4.39)$$

B:

$$\sum_{n=-N}^{N} n^j h_n = 0, \quad j = 1,2, \ldots, p. \qquad (4.40)$$

The transfer function of a digital smoothing filter which approximates smoothing filters of the types discussed in section 4.2 is an even function of f and can be written in the form

$$H_n(f) = h_o + 2 \sum_{n=1}^{N} h_n \cos 2n\pi \frac{f}{f_s}. \qquad (4.41)$$

The weights are related by $h_n = h_{-n}$. Hence for odd integers j,

$$n^j h_n = -(-n)^j h_{-n}$$

or

$$n^j h_n + (-n)^j h_{-n} = 0$$

and

$$\sum_{n=-N}^{N} n^j h_n = 0. \tag{4.42}$$

Thus (4.40) is satisfied for all odd integers j without imposing any conditions on the h_n. If (4.39) is satisfied, the filter passes a first degree polynomial exactly. If, in addition, (4.40) is satisfied for j=2, the filter passes a third degree polynomial exactly, etc. Practical considerations usually limit j to 2, i.e., p=3.

The simplest way to satisfy (4.39) is to use new weights

$$\bar{h}_n = \frac{h_n}{\sum_{n=-N}^{N} h_n}. \tag{4.43}$$

If N is chosen so that $.005 \leq \epsilon \leq .01$, the new weights usually do not change ϵ significantly.

For $j \geq 2$ the usual approach is to derive the constrained weights $\bar{h}_n$ so that the mean square error between the unconstrained transfer function $H_N(f)$ and the constrained transfer function

$$\bar{H}_N(f) = \bar{h}_o + 2 \sum_{n=0}^{N} \bar{h}_n \cos 2n\pi \frac{f}{f_s} \tag{4.44}$$

is minimized.

Note that (4.39) is equivalent to the condition

$$\bar{H}_N(0) = 1, \tag{4.45}$$

and (4.40) is equivalent to the conditions

$$\left. \frac{d^j \bar{H}_N(f)}{df^j} \right|_{f=0} = 0, \quad 1 \leq j \leq p. \tag{4.46}$$

Taking the case p=3 and using a Lagrangian multiplier, we wish to find weights $\bar{h}_n$ in terms of the h_n such that

$$R = \int_0^{\frac{f_s}{2}} [\bar{H}_N(f) - H_N(f)]^2 \, df + \lambda \sum_{n=1}^{N} n^2 \bar{h}_n$$

is minimized, i.e., $\frac{\partial R}{\partial \bar{h}_m} = 0$, $0 \le m \le N$, and such that $\bar{H}_N(f)$ satisfies (4.45) and (4.46) for p=3.

$$\frac{\partial R}{\partial \bar{h}_m} = 2 \int_0^{\frac{f_s}{2}} [\bar{H}_N(f) - H_N(f)] \frac{\partial \bar{H}_N(f)}{\partial \bar{h}_m} \, df + \lambda m^2. \tag{4.47}$$

The condition (4.45) is incorporated in the following way:

$$\bar{H}_N(0) = \bar{h}_o + 2 \sum_{n=1}^{N} \bar{h}_n = 1,$$

so

$$\bar{h}_o = 1 - 2 \sum_{n=1}^{N} \bar{h}_n. \tag{4.48}$$

Hence

$$\bar{H}_N(f) - H_N(f) = 1 + 2 \sum_{n=2}^{N} \bar{h}_n (\cos 2n\pi \frac{f}{f_s} - 1) - H_N(f).$$

Therefore

$$\frac{\partial \bar{H}_N}{\partial \bar{h}_m} = 2 \, (\cos 2m\pi \frac{f}{f_s} - 1),$$

and

$$\frac{\partial R}{\partial \bar{h}_m} = 4\int_0^{\frac{f_s}{2}} [1 + 2\sum_{n=1}^{N} \bar{h}_n(\cos 2n\pi\frac{f}{f_s} - 1) - h_o - 2\sum_{n=1}^{N} h_n \cos 2n\pi\frac{f}{f_s}]$$

$$[\cos 2m\pi\frac{f}{f_s} - 1]\, df \quad + \lambda m^2.$$

Let $\theta = 2\pi \frac{f}{f_s}$, then $df = \frac{f_s\, d\theta}{2\pi}$ and

$$\frac{\partial R}{\partial \bar{h}_m} = \frac{2f_s}{\pi} \int_0^{\pi} [1 - 2\sum_{n=1}^{N}\bar{h}_n - h_o + 2\sum_{n=1}^{N} (\bar{h}_n - h_n)\cos n\theta][\cos m\theta - 1]d\theta + \lambda m^2.$$

$$= \frac{2f_s}{\pi}\int_0^{\pi}\Big\{[1 - 2\sum_{n=1}^{N}\bar{h}_n - h_o][\cos m\theta - 1] + 2\sum_{n=1}^{N}(\bar{h}_n - h_n) \cos n\theta \cos m\theta$$

$$-2\sum_{n=1}^{N}(\bar{h}_n - h_n) \cos n\theta\Big\} \; d\theta + \lambda m^2$$

$$= \frac{2f_s}{\pi} \Big\{-\pi[1 - 2\sum_{n=1}^{N}\bar{h}_n - h_o] + \pi (\bar{h}_m - h_m)\Big\} + \lambda m^2.$$

Setting this equal to zero gives

$$2f_s\Big\{[h_o - 1 + 2\sum_{n=1}^{N}\bar{h}_n + \bar{h}_m - h_m]\Big\} + \lambda m^2 = 0.$$

From (4.48) we see that we can replace $2\sum_{n=1}^{N}\bar{h}_n - 1$ by $-\bar{h}_o$.

Thus

$$2f_s\left\{[h_o-\bar{h}_o + \bar{h}_m-h_m]\right\} + \lambda m^2 = 0.$$

Let $\delta = \bar{h}_o-h_o$, then

$$\bar{h}_m-h_m = \delta - \frac{\lambda m^2}{2f_s}. \tag{4.49}$$

Summing both sides of (4.49) from 1 to N, then multiplying both sides by 2, and adding δ to both sides gives

$$\delta + 2\sum_{m=1}^{N}\bar{h}_m - 2\sum_{m=1}^{N}h_m = (2N+1)\,\delta - \frac{\lambda}{f_s}\sum_{m=1}^{N}m^2$$

or using (4.45) and reverting to the n subscript,

$$(2N+1)\,\delta - \frac{\lambda}{f_s}\sum_{n=1}^{N}n^2 = 1 - h_o - 2\sum_{n=1}^{N}h_n. \tag{4.50}$$

Multiplying both sides of (4.49) by m^2, summing from 1 to N, using (4.46)--(or 4.40 with j=2)--and reverting to the n subscript gives

$$\delta\sum_{n=1}^{N}n^2 - \frac{\lambda}{2f_s}\sum_{n=1}^{N}n^4 = -\sum_{n=1}^{N}n^2h_n \tag{4.51}$$

We solve (4.50) and (4.51) for δ and λ.

Let

$$Q_1 = 1 - h_o - 2\sum_{n=1}^{N}h_n$$

$$Q_2 = \sum_{n=1}^{N}n^2h_n$$

$$S_1 = \sum_{n=1}^{N} n^2$$

$$S_2 = \sum_{n=1}^{N} n^4 .$$

Then

$$\lambda = \frac{2f_s[S_1Q_1+(2N+1)Q_2]}{(2N+1)S_2-2S_1^{\,2}} \tag{4.52}$$

and

$$\delta = \frac{Q_1S_2+2S_1Q_2}{(2N+1)S_2-2S_1^{\,2}} . \tag{4.53}$$

Then

$$\bar{h}_o = h_o + \delta \tag{4.54a}$$

and from (4.49), for $n \geq 1$

$$\bar{h}_n = h_n + \delta - \frac{n^2}{2f_s}\lambda \tag{4.54b}$$

$$= h_n + \frac{Q_1S_2+2S_1Q_2-n^2[S_1Q_1+(2N+1)Q_2]}{(2N+1)S_2-2S_1^{\,2}} .$$

Note that

$$(2N+1)S_2-2S_1^{\,2} = \frac{N(N+1)(2N-1)(2N+3)(2N+1)^2}{90}.$$

The constraint for p=1 is obtained by letting $\lambda = 0$ in (4.54b). Then we have

$$\bar{h}_n = h_n + \delta , \qquad n = 0,1, \ldots, N$$

where

$$\delta = \frac{1-h_o-2\sum_{n=1}^{N} h_n}{2N+1}.$$

$$= \frac{1-H_N(0)}{2N+1}.$$

4.7 Band-pass filter

The ideal band-pass smoothing filter transfer function is

$$B_I(f) = \begin{cases} 1 & f_c \le f \le \bar{f}_c \\ 0 & 0 \le f < f_c,\ f > \bar{f}_c. \\ B_I(-f) & f < 0 \end{cases} \qquad (4.55)$$

See figure 4.7.

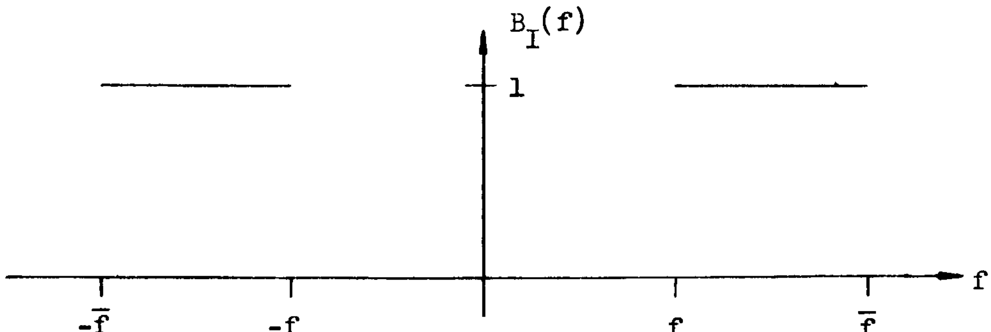


Figure 4.7.

Note that $B_I(f)$ can be written as the difference of two ideal smoothing filter transfer functions [see (3.7)] $H_1(f)$ and $H_2(f)$, where $H_2(f)$ has cut-off $\bar{f}_c$ and $H_1(f)$ has cut-off f_c. Then the weight function $b(t)$ is

$$b(t) = h_2(t) - h_1(t) \qquad (4.56)$$

where

$$h_1(t) \longleftrightarrow H_1(f),$$

$$h_2(t) \longleftrightarrow H_2(f),$$

and

$$b(t) \longleftrightarrow B_1(f).$$

A useable design is obtained by taking the difference of two low-pass smoothing filters of the types discussed in section 4.2 and section 4.4. The difference of two Martin-Graham filters, each with roll-off length Δf gives a satisfactory filter. The weight function of the resulting band-pass filter is then given by (4.56) with $h_1(t)$ and $h_2(t)$ the weights of the Martin-Graham filters. The weights of the corresponding digital filter are given by (3.44) and (4.56),

$$\begin{aligned} b_n &= \frac{1}{f_s}\, b\left(\frac{-n}{f_s}\right) \\ &= \frac{1}{f_s}\left[h_2\left(\frac{-n}{f_s}\right) - h_1\left(\frac{-n}{f_s}\right)\right]. \end{aligned} \tag{4.57}$$

Now suppose $B(f;f_o)$ is a band-pass smoothing filter with the mid-points of the "pass bands" at $\pm\, f_o$, "pass band" width $2\overline{\Delta f}$, and roll-off width Δf. For purposes of illustration, we assume that $B(f;f_o)$ has the Martin-Graham type roll-off [see (4.18)]. Let

$$H(f) = \begin{cases} 1 & 0 \leq f \leq \overline{\Delta f} \\ \frac{1}{2}\left[1 + \cos\frac{(f-\overline{\Delta f})\pi}{\Delta f}\right] & \overline{\Delta f} < f \leq \overline{\Delta f}+\Delta f \\ 0 & f > \overline{\Delta f}+\Delta f \\ H(-f) & f < 0\,. \end{cases} \tag{4.58}$$

See figure 4.8.

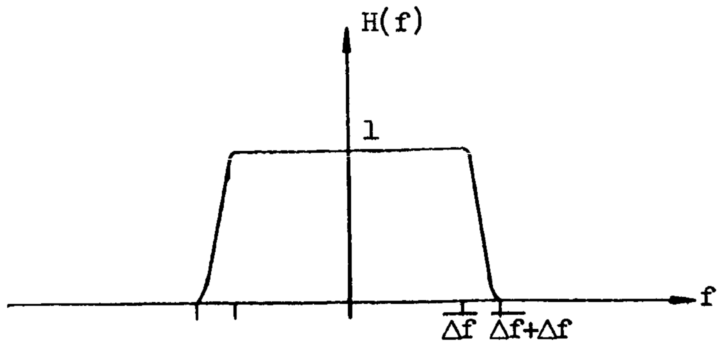


Figure 4.8.

Then H(f) is the transform of

$$h(t) = \frac{\sin 2\pi\overline{\Delta f} + \sin 2\pi(\Delta f+\overline{\Delta f})t}{2\pi t(1 - 4\Delta f^2 t^2)} . \tag{4.59}$$

For $f \geq 0$

$$B(f;f_o) = H(f - f_o),$$

and for $f < 0$

$$B(f;f_o) = H(f + f_o).$$

Thus

$$B(f;f_o) = H(f - f_o)+H(f + f_o), \tag{4.60}$$

see figure 4.9.

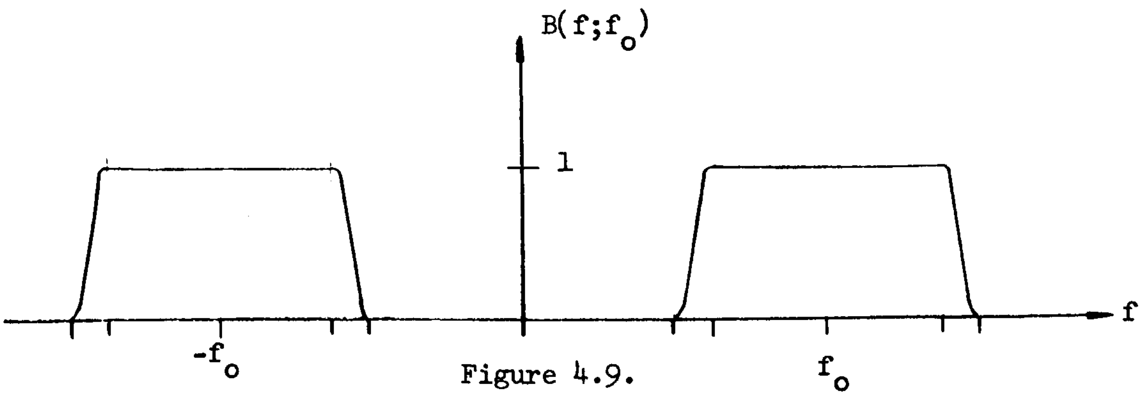


Figure 4.9.

Taking the inverse transform of each side and using the shift theorem (1.21) gives

$$b(t;t_o) = h(t)(e^{2\pi i f_o t} + e^{-2\pi i f_o t})$$

$$= 2h(t)\cos 2\pi f_o t. \tag{4.61}$$

The weights of the corresponding digital filter are given by

$$b_n(f_o) = 2h_n \cos 2\pi n \frac{f_o}{f_s}, \tag{4.62}$$

where $h_n = \frac{1}{f_s} h(\frac{-n}{f_s})$.

For a given f_o, the weights can be computed from (4.62) more quickly than from (4.57). If several successive filtering operations are to be performed for a set of f_o values, say $f_1, f_2, \ldots, f_k$, then, using (4.62),

$$b_n(f_j) = 2h_n \cos 2n\pi \frac{f_j}{f_s}, \qquad j = 1,2, \ldots, k.$$

But in order to use (4.57) the functions $h_1(t)$ and $h_2(t)$ must be changed for each new value of f_j and the entire expression must be recomputed.

From (4.62) we see that the error ϵ' of a band-pass smoothing filter may be as much as twice the error ϵ of the smoothing filter whose transfer function is $H(f)$.

In a manner analogous to the ideal smoothing case in section 3.2, the transfer function of a filter which will simultaneously "band-pass" filter and find the $n^{\underline{th}}$ derivative is

$$B^n(f) = (2\pi i f)^n B(f) \tag{4.63}$$

where B(f) is the transfer function of a band-pass smoothing filter. Then if $b(t) \longleftrightarrow B(f)$,

$$b^{(n)}(t) = b^n(t) \longleftrightarrow B^n(f), \tag{4.64}$$

and the weights [see the derivation of (4.3b)] are given by

$$b_k^n = (-1)^n (f_s)^n \frac{d^n b_k}{dk^n}, \tag{4.65}$$

where $b_k = \frac{1}{f_s} b(\frac{-k}{f_s})$.

The weight function of a filter having several pass bands, each of equal pass width and roll-off width, can easily be found from (4.61). Let $\pm f_1, \pm f_2, \ldots, \pm f_k$ be the mid points of the pass bands, and denote the transfer function by $B(f;f_1,f_2,\ldots,f_k)$ (see figure 4.10).

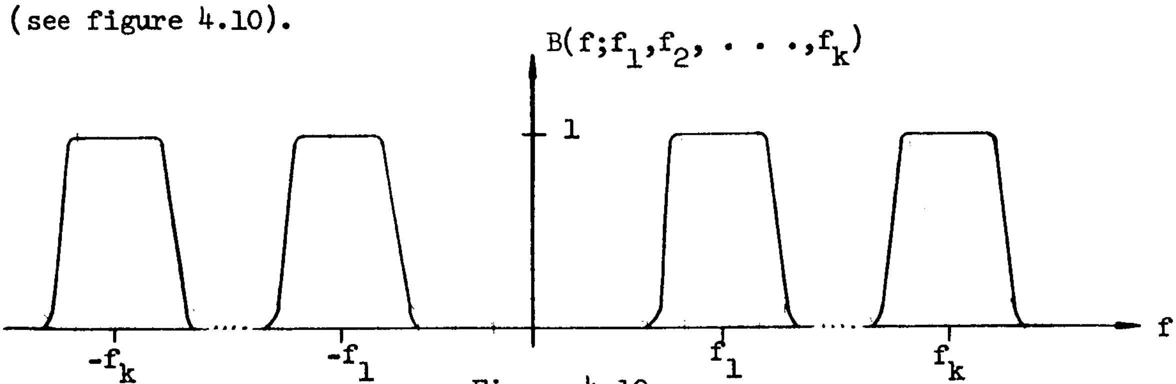


Figure 4.10.

Then the weight function is

$$b(t;f_1,f_2,\ldots,f_k) = 2\left(\sum_{j=1}^{k} \cos 2\pi f_j t\right) h(t). \tag{4.66}$$

The weights are given by

$$b_n = \frac{1}{f_s}\, b(\frac{-n}{f_s};\ f_1, f_2, \ . \ . \ . , f_k)$$

$$= 2h_n \left(\sum_{j=1}^{k} \cos 2n\pi \frac{f_j}{f_s} \right). \qquad (4.67)$$

CHAPTER V

MARTIN-GRAHAM FILTERS

5.1 Introduction

We shall call a filter a Martin-Graham filter if its transfer function either uses the Martin-Graham roll-off [see (4.18)] or is derivable from a transfer function having the Martin-Graham roll-off.

In section 4.2, we discussed the Martin-Graham smoothing filter and found its weight function h(t) [see (4.13)]. From h(t) and the formula (3.44) for computing the weights of the approximating digital filter, we found the weights h_n [see(4.15)] which are used in the basic formula of digital filtering,

$$\bar{f}_m = \sum_{n=-N}^{N} h_n g_{m+n}, \tag{3.45}$$

where the g_j are the input data values and the $\bar{f}_j$ are the smoothed output values. A simple but detailed example of an application is given in Chapter VII.

A Martin-Graham band-pass smoothing filter is easily obtained from the smoothing case and the discussion of section 4.7.

In this chapter, we shall derive the weights for some other Martin-Graham filters. When referring to a set of data $\left\{g_m\right\}$, we assume that the data arises from a function g(t) such that

1) $g(t) = \bar{g}(t) + p(t)$, where p(t) is a polynomial in t,

2) $\bar{g}(t)$ satisfies conditions I - III of section 4.1,

3) $g_m = g(\frac{m}{f_s})$ where f_s is greater than twice the highest frequency in $\bar{g}(t)$.

Let g_M be the first data value and $g_{\overline{M}}$ be the last. If p(t) is not identically zero for $\frac{M}{f_s} \leq t \leq \frac{\overline{M}}{f_s}$, then, in order to pass p(t) or differentiate it, constraints are necessary. Those for smoothing are in section 4.6. A general procedure is given in appendix A for the derivative cases, and the constraints for passing the first derivative of p(t) will be given in the next section.

5.2 Smoothing and first derivative filter

We have shown that the transfer function of a filter which will smooth and find the first derivative to be

$$Y^1(w) = iwH(w)$$

where H(w) is any smoothing filter transfer function [Put n=1 and $w = 2\pi f$ in (4.2)]. Note that $Y^1(w)$ inherits the cut-off, w_c, and termination, w_T, frequencies from H(w).

Putting n=1 in (4.3b), we obtain the weights of this filter in terms of the smoothing weights

$$y_k^1 = -f_s \frac{dh_k}{dk} \tag{5.1}$$

where $h(t) \longleftrightarrow H(w)$ and $h_k = \frac{1}{f_s} h(\frac{-k}{f_s})$.

The Martin-Graham smoothing filter weights given by (4.15) in terms of the frequency ratio, $r = \frac{w}{2\pi f_s} = \frac{f}{f_s}$, are

$$h_k = \frac{\cos k\pi r_d \sin k\pi(2r_c + r_d)}{k\pi(1-4r_d^2k^2)}$$

$$= \frac{\sin 2\pi r_T k + \sin 2\pi r_c k}{2\pi k(1-4r_d^2k^2)},$$

$$r_d = \frac{\Delta f}{f_s},\ r_c = \frac{f_c}{f_s},\ r_T = \frac{f_T}{f_s}.$$

Then

$$y_k^1 = -f_s \quad \frac{r_T \cos 2\pi r_T k + r_c \cos 2\pi r_c k}{k(1-4r_d{}^2k^2)} - \frac{h_k(1-12r_d{}^2k^2)}{k(1-4r_d{}^2k^2)}$$

$$= -f_s \quad \frac{r_T \cos 2\pi r_T k + r_c \cos 2\pi r_c k - h_k(1-12r_d{}^2k^2)}{k(1-4r_d{}^2k^2)}. \qquad (5.2)$$

Note that $y_{-k}^1 = -y_k^1$, and by applying L'Hospital's rule, $y_0^1 = 0$.

In a manner analogous to that of section 4.6, we find that in order to pass exactly the derivative of P(t) of degree p the following conditions must be satisfied by the approximating filter transfer function

$$Y_N^1(w) = 2i \sum_{n=1}^{N} y_n^1 \sin \frac{wn}{f_s}. \qquad (5.3)$$

(1) $Y_N^1(0) = 0$

(2) $\left.\dfrac{dy_N^1(w)}{dw}\right|_{w=0} = i$

(3) $\dfrac{d^p Y_N^1(w)}{dw^p} = 0 \quad \text{for } p > 1.$

Since $\dfrac{d^p Y_N^1(w)}{dw^p}$ is odd for all even $p \geq 0$, (1) and (3) are automatically satisfied for even integers $p \geq 0$. In particular, if p(t) is of degree 2, we need to satisfy only (2). The constrained weights $\bar{y}_k^1$ are given by

$$\bar{y}_k^1 = y_k^1 + \frac{kQ_1}{Q_2}, \qquad k \geq 1, \qquad (5.4)$$

$$\bar{y}_k^1 = -\bar{y}_k^1,$$

where

$$Q_1 = \frac{f_s}{2} - \sum_{n=1}^{N} n\overset{1}{y}_n$$

$$Q_2 = \sum_{n=1}^{N} n^2 .$$

(See [11] for the derivation for p=4 from which the case p=2 follows easily.)

The constrained transfer function is

$$\overset{1}{\overline{Y}}_N(w) = 2i \sum_{n=1}^{N} \overset{1}{\overline{y}}_n \sin \frac{wn}{f_s} . \tag{5.5}$$

In order to smooth and differentiate a set of data $_1\{g_m\}$ where the polynomial content is of degree 2 or less, put $h_n = \overline{y}_n$ in (3.45). This gives

$$\overline{f}_m = \sum_{n=-N}^{N} \overset{1}{\overline{y}}_n \, g_{m+n} . \tag{5.6}$$

If we let $\hat{h}_k = \frac{h_k}{f_s}$, then $\overset{\wedge 1}{y}_k = \frac{\overset{1}{y}_k}{f_s}$ and $\overset{\wedge 1}{\overline{y}}_k = \frac{\overset{1}{\overline{y}}_k}{f_s}$. Then

$$\overset{1}{\overline{Y}}_N(w) = 2if_s \sum_{n=1}^{N} \overset{\wedge 1}{\overline{y}}_k \sin \frac{wn}{f_s} \tag{5.7}$$

and

$$\overline{f}_m = f_s \sum_{n=-N}^{N} \overset{\wedge 1}{\overline{y}}_n \, g_{m+n} . \tag{5.8}$$

Using $\hat{y}^1_{-n} = -\hat{y}^1_n$, we have

$$\bar{f}_m = f_s \sum_{n=1}^{N} \hat{y}^1_n \, [g_{m+n} - g_{m-n}]. \tag{5.9}$$

Writing

$$Y^1(w) = iwH(w) = wH(w)e^{\frac{i\pi}{2}},$$

we see that $Y^1(w)$ has a phase shift of 90^o. $wH(w)$ is shown in figure 5.1 for the Martin-Graham smoothing and first derivative filter.

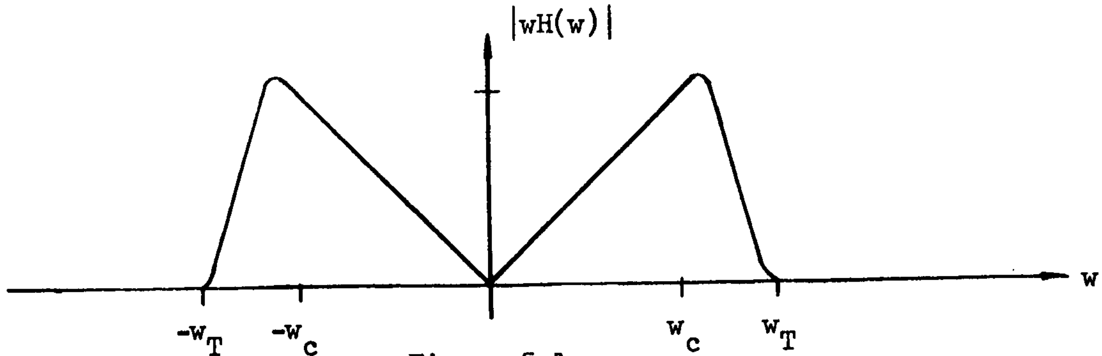


Figure 5.1.

Other first derivative filters with different roll-offs have been examined and it was found that the Martin-Graham filter yielded the same or a more accurate result.

In an attempt to avoid the lengthy computation of (5.1) for the weights y^1_k, a "three-point derivative" of the smoothing weights h_k has been examined. Let

$$\theta_k = \frac{h_{k+1} - h_{k-1}}{\frac{2}{f_s}}. \tag{5.10}$$

With H(f) the transform of the weight function h(t) from which the h_k are computed, we have

$$\theta_k = \frac{f_s}{2}\,[h_{k+1} - h_{k-1}]$$

$$= \frac{1}{2}\left[h\left(\frac{-k-1}{f_s}\right) - h\left(\frac{-k+1}{f_s}\right)\right]$$

$$= \frac{1}{2}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} H(f)\left[e^{-2\pi i(k+1)\frac{f}{f_s}} - e^{-2\pi i(k-1)\frac{f}{f_s}}\right]df$$

$$= \frac{1}{2}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} H(f)e^{-2\pi ik\frac{f}{f_s}}\left[e^{-2\pi i\frac{f}{f_s}} - e^{2\pi i\frac{f}{f_s}}\right]df$$

$$= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \left[\frac{e^{2\pi i\frac{f}{f_s}} - e^{-2\pi i\frac{f}{f_s}}}{2i}\right]\left[-iH(f)e^{-2\pi ik\frac{f}{f_s}}\right]df$$

$$\theta_k = \frac{1}{f_s}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \frac{-\sin 2\pi\frac{f}{f_s}}{2\pi\frac{f}{f_s}}\left[(2\pi if)H(f)\right]e^{-2\pi ik\frac{f}{f_s}}df. \qquad (5.11)$$

The actual weights are

$$y_k^1 = \frac{1}{f_s}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} (2\pi if)H(f)e^{-2\pi ik\frac{f}{f_s}}\,df\ . \qquad (5.12)$$

Comparing (5.11) and (5.12), we see that if we define a weight $\tilde{y}_k = -\theta_k$, then the transfer function of the $\tilde{y}_k$ is

$$\frac{\sin 2\pi\frac{f}{f_s}}{2\pi\frac{f}{f_s}}(2\pi i f)H(f)$$

which is the product of the desired transfer function and

$$\frac{\sin 2\pi\frac{f}{f_s}}{2\pi\frac{f}{f_s}}.$$

Now

$$y_k^1 - \tilde{y}_k = \frac{1}{f_s}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}}(2\pi i f)H(f)\left[1 - \frac{\sin 2\pi\frac{f}{f_s}}{2\pi\frac{f}{f_s}}\right]e^{-2\pi i k\frac{f}{f_s}}\,df$$

and

$$1 - \frac{\sin 2\pi\frac{f}{f_s}}{2\pi\frac{f}{f_s}} \doteq 0$$

for $\left|\frac{f}{f_s}\right|$ small. If the cut-off f_c is small, then $H(f)$ in the above integral becomes zero for $\frac{f}{f_s}$ relatively small. Then the $\tilde{y}_k$ are good approximations of the y_k^1. It has been found empirically that for filters such that $\frac{f_c}{f_s} \leq .1$, the $\tilde{y}_k$ give an acceptable output.

5.3 Band-pass smoothing and first derivative filter

We have shown that the transfer function of a band-pass filter which will smooth and find the first derivative to be

$$B^1(w) = iwB(w)$$

where B(w) is any band-pass smoothing filter transfer function [put n=1 and w=2πf in (4.63)]. Note that $B^1(w)$ has the same cutoff and termination frequencies as B(w). B(w) may be designed by either of the methods discussed in section 4.7.

Putting n=1 in (4.65), we obtain the weights of this filter in terms of the band-pass smoothing weights

$$b_k^1 = -f_s \frac{db_k}{d_k}$$

where $b(t) \longleftrightarrow B(w)$ and $b_k = \frac{1}{f_s} b(\frac{-k}{f_s})$.

If the b_k are obtained by taking the difference [see (4.57)] of the weights of two low-pass filters, say h_k' and h_k'', then

$$b_k^1 = -f_s \left\{\frac{dh_k''}{dk} - \frac{dh_k'}{dk}\right\} . \qquad (5.13)$$

When the b_k are obtained by the second method [see (4.62)], we have

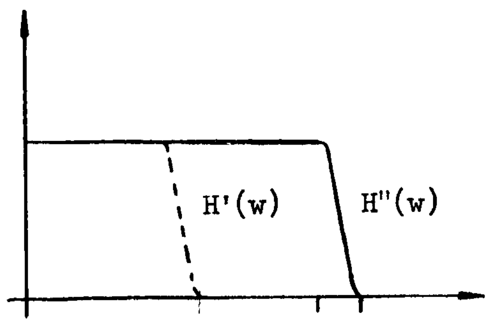
$$b_k^1 = -2f_s \frac{d}{dk} \left\{ h_k \cos 2k\pi\frac{f_o}{f_s} \right\} \qquad (5.14)$$

$$= -2f_s \left\{ \frac{-h_k 2\pi f_o}{f_s} \sin 2k\pi\frac{f_o}{f_s} + \frac{dh_k}{dk} \cos 2k\pi\frac{f_o}{f_s} \right\}$$

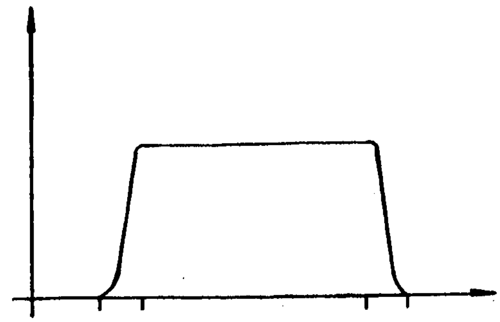
$$= 4\pi h_k f_o \sin 2k\pi\frac{f_o}{f_s} - 2f_s \frac{dh_k}{dk} \cos 2k\pi\frac{f_o}{f_s} .$$

To obtain a Martin-Graham filter of this type by the first method, we simply select two Martin-Graham smoothing filters with transfer-weight functions $h'(t) \longleftrightarrow H'(w)$ and $h''(t) \longleftrightarrow H''(w)$ and compute the weights b_k^1 by (5.13). To use the second method, the appropriate

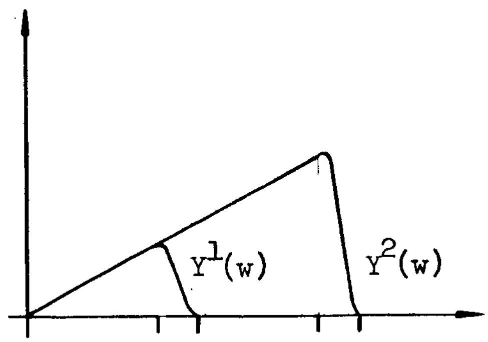
Martin-Graham filter with $h(t) \longleftrightarrow H(w)$ is selected and the weights b_k^1 are computed by (5.14). These weights are used for the h_k in (3.45). Note that a factor of f_s can be removed from the sum (3.45) in a manner analogous to the first derivative case [see (5.7) and (5.8)].



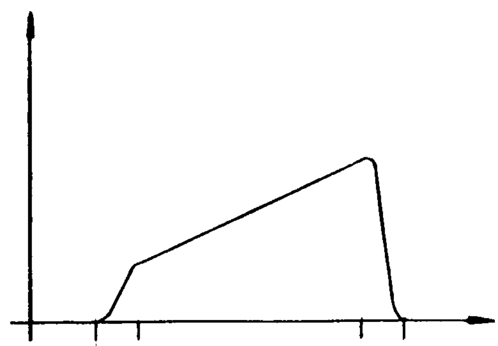
(a) Smoothing transfer functions



(b) Band-pass: $H''(w) - H'(w)$



(c) Derivative transfer functions derived from H' and H''.



(d) Band-pass derivative: $Y^2(w) - Y^1(w)$

Figure 5.2.

5.4 Smoothing and second derivative filter.

Letting n=2 and $w=2\pi f$ in (4.2), we find that the transfer function of a filter which will smooth and find the second derivative is

$$Y^2(w) = -w^2 H(w) \tag{5.15}$$

where H(w) is any smoothing filter transfer function.

Putting n=2 in (4.3b), we find that the weights of the filter in terms of the smoothing weights are

$$y_k^2 = f_s^2 \frac{d^2 h_k}{dk^2} \tag{5.16}$$

where $h(t) \longleftrightarrow H(w)$ and $h_k = \frac{1}{f_s} h(\frac{-k}{f_s})$.

Using the Martin-Graham smoothing weights given by (4.15) in terms of $r = \frac{w}{2\pi f_s} = \frac{f}{f_s}$ and (5.16) gives

$$y_k^2 = \frac{f_s^2}{k(1-4r_d^2k^2)} [24r_d^2 k h_k - \frac{2y_k^1}{f_s}(1-12r_d^2k^2) - 2\pi(r_T^2 \sin 2\pi r_T k + r_c^2 \sin 2\pi r_c k)] \tag{5.17}$$

where y_k^1 is given by equation (5.2).

For k=0, using L'Hospital's rule gives

$$y_o^2 = f_s^2 \left\{8(r_T+r_c) - \frac{4\pi^2}{3}(r_T^3+r_c^3)\right\} . \tag{5.18}$$

This gives the weights to be used in the formula (3.45). Note that a factor of f_s^2 may be removed in this case.

A constraint is developed in appendix A to improve the fit of the approximating transfer function at some specific frequency ratio $\bar{r}$.

CHAPTER VI

INTEGRATION

6.1 The Romberg technique.

One approach to the problem of integrating a set of numerical data is to smooth the input data and then apply one of the standard numerical integration techniques. A recently developed technique which seems to be well suited to the smoothed output of a filter is the one developed by Romberg (see appendix B). If

$$I = \int_a^{a+2^k\Delta t} f(s)ds \tag{6.1}$$

where $f(s)$ is known at the 2^k+1 points: $a, a+j\,\Delta t$, $j=1,2, \ldots, 2^k$; $\Delta t = \frac{1}{f_s}$, then we approximate I by the diagonal element in the $k^{\underline{th}}$ row of the Romberg array, i.e.,

$$I \triangleq 2^k \Delta t T_{k,0}. \tag{6.2}$$

6.2 Integrating filters.

Another approach to the problem more in keeping with the one of this report is to design a filter which will simultaneously smooth and either give the indefinite integral or a definite integral of the input function.

Let $Ae^{2\pi ift}$ be a component of an input to a filter. Assuming that the constant of integration is zero, the indefinite integral of this component is $(2\pi if)^{-1}Ae^{2\pi ift}$. If this is to be the output of the filter, then, using (3.5), we find that the transfer function must be

$$x(f) = (2\pi if)^{-1} \tag{6.3}$$

Letting $k'(t) = g(t)$ and assuming that the constant of integration is zero and that $k(t)$ satisfies conditions sufficient for the Fourier integral theorem to hold, we have

$$k(t) \longleftrightarrow (2\pi i f)^{-1} G(f) \tag{6.4}$$

where $g(t) \longleftrightarrow G(f)$.

If we also smooth, we have

$$\bar{k}(t) \longleftrightarrow (2\pi i f)^{-1} H(f) G(f) \tag{6.5}$$

where $H(f)$ is the smoothing filter transfer function. Then the transfer function of a filter which will simultaneously smooth and give the indefinite integral is

$$Y^{(-1)}(f) = (2\pi i f)^{-1} H(f). \tag{6.6}$$

Note that the smoothed output, $\bar{g}(t)$, of the smoothing filter is the inverse transform of $H(f)G(f)$ and that

$$\bar{k}(t) = \int \bar{g}(\alpha) d\alpha. \tag{6.7}$$

For the transfer functions, $H_j(f)$, $j=1,2, \ldots, 5$, of the smoothing filters discussed in Chapter IV, $Y^{(-1)}(f)$ has an infinite discontinuity at $f=0$. Hence, in order to approximate $Y^{(-1)}(f)$ with a truncated Fourier series, we must modify $Y^{(-1)}(f)$ on an interval containing zero. To avoid some integrals which cannot be evaluated in closed form, we shall consider only the case $j=1$, i.e., an Ormsby type filter.

Let $\Delta f > 0$ and

$$Y^{(-1)}(f) = (2\pi i)^{-1} \begin{cases} f(\Delta f)^{-2} , & |f| < \Delta f , \\ f^{-1} , & \Delta f \leq f \leq f_c , \\ \dfrac{f_T - f}{f_c \Delta f} , & f_c < f \leq f_T , \\ 0 , & |f| > f_T, \end{cases} \tag{6.8}$$

and $Y^{(-1)}(-f) = -Y^{(-1)}(f)$ for $f < 0$. See figure 6.2.

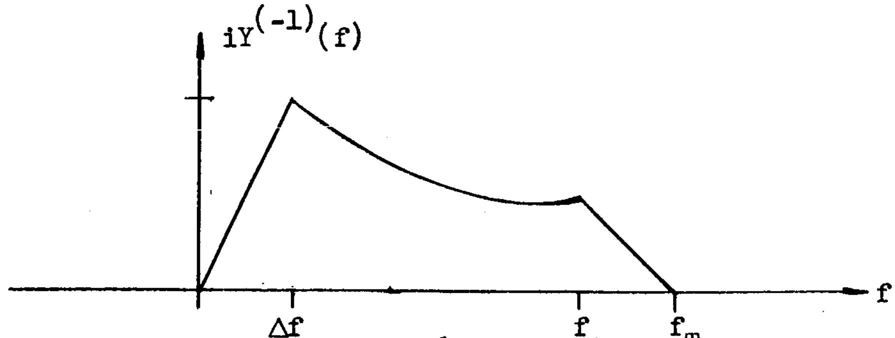


Figure 6.2.

The weights in terms of the frequency ratio $r = \frac{f}{f_s}$, $r_d = \frac{\Delta f}{f_s}$, $r_c = \frac{f_c}{f_s}$, $r_T = \frac{f_T}{f_s}$, are

$$y_n^{(-1)} = \frac{1}{2\pi^2 r_d f_s} \left\{ \frac{\cos 2n\pi r_d}{n} - \frac{\sin 2n\pi r_d}{2\pi r_d n^2} \right.$$

$$- \frac{r_d \cos 2n\pi r_c}{r_c n} + \frac{(\sin 2n\pi r_T - \sin 2n\pi r_c)}{2\pi r_c n^2}$$

$$\left. - 2\pi r_d \left[\mathrm{Si}\,(2n\pi r_c) - \mathrm{Si}(2n\pi r_d)\right] \right\}, \tag{6.9}$$

where

$$\mathrm{Si}(x) = \int_0^x \frac{\sin y}{y}\, dy = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{(2k-1)}}{(2k-1)!\,(2k-1)} .$$

Define $Y_0^{(-1)} = 0$. Also, $Y_{-k}^{(-1)} = -Y_k^{(-1)}$

<u>A definite integral</u>

From (6.7) we have

$$\bar{k}(t+a) - \bar{k}(t-a) = \int_{t-a}^{t+a} \bar{g}(\alpha)\, d\alpha,$$

and by (6.5) and the shift theorem,

$$\bar{k}(t+a) - \bar{k}(t-a) \longleftrightarrow (2\pi if)^{-1}G(f)H(f)[e^{2\pi iaf} - e^{-2\pi iaf}]$$

or

$$\bar{k}(t+a) - \bar{k}(t-a) \longleftrightarrow \frac{\sin 2\pi af}{\pi f} H(f)G(f). \tag{6.10}$$

Thus, if H(f) is a smoothing filter transfer function, the transfer function of a filter which will simultaneously smooth and give the integral of the input over [t-a,t+a] is

$$Y^{(-1)}(f) = \frac{\sin 2\pi af}{\pi f} H(f). \tag{6.11}$$

Let

$$x(t) = \begin{cases} 1 & |t| \leq a \\ 0 & |t| > a \end{cases}, \tag{6.12}$$

then

$$x(t) \longleftrightarrow X(f) = \frac{\sin 2\pi af}{\pi f}. \tag{6.13}$$

Applying the convolution theorem gives

$$\begin{aligned} y^{(-1)}(t) &= \int_{-\infty}^{\infty} h(z)x(t-z)dz \\ &= \int_{t-a}^{t+a} h(z)dz \\ &= \int_{-a}^{a} h(t-z)dz \end{aligned} \tag{6.14}$$

where $y^{(-1)}(t) \longleftrightarrow Y^{(-1)}(f)$ and $h(t) \longleftrightarrow H(f)$.

By (3.44), the weights are

$$y_n^{(-1)} = \frac{1}{f_s} \int_{\frac{-n}{f_s}-a}^{\frac{-n}{f_s}+a} h(z)\,dz. \tag{6.15}$$

Choosing $h_1(t)$, the Ormsby smoothing filter weight function, we have

$$y^{(-1)}(t) = \frac{1}{\pi\Delta f}\left\{\frac{t(\sin 2\pi f_c t \sin 2\pi f_c a-\sin 2\pi f_T t \sin 2\pi f_T a)}{\pi(t^2-a^2)}\right.$$

$$+ \frac{a(\cos 2\pi f_c t \cos 2\pi f_c a - \cos 2\pi f_T t \cos 2\pi f_T a)}{\pi(t^2-a^2)} \tag{6.16}$$

$$- f_c[\mathrm{Si}(2\pi f_c[t+a]) - \mathrm{Si}(2\pi f_c[t-a])]$$

$$\left. + f_T[\mathrm{Si}(2\pi f_T[t+a]) - \mathrm{Si}(2\pi f_T[t-a])]\right\}.$$

Using the frequency ratio $r = \frac{f}{f_s}$, letting $a = \frac{b}{f_s}$, and computing the weights by (3.44), we have

$$y_n^{(-1)} = \frac{1}{\pi^2 r_d f_s}\left\{\frac{n(\sin 2n\pi r_c \sin 2\pi r_c b - \sin 2n\pi r_T \sin 2\pi r_T b)}{(n^2-b^2)}\right.$$

$$+ \frac{b(\cos 2n\pi r_c \cos 2\pi r_c b - \cos 2n\pi r_T \cos 2\pi r_T b)}{(n^2-b^2)}$$

$$- \pi r_c[\mathrm{Si}(2\pi r_c[n+b]) - \mathrm{Si}(2\pi r_c[n-b])]$$

$$\left. + \pi r_T[\mathrm{Si}(2\pi r_T[n+b]) - \mathrm{Si}(2\pi r_T[n-b])]\right\}.$$

CHAPTER VII

APPLICATIONS

7.1 Editing and determination of digital filter parameters.

In order to apply a digital filter to a set of data $\{g_m\}$, we assume that the data values are obtained by taking equally spaced samples of a function g(t) which satisfies the three conditions of section 5.1. A variety of problems may arise from the methods used to obtain the samples, and editing may be necessary. Common problems are missing values and "bad" values, i.e., values grossly in error. Since these can affect the output considerably, it is important to replace them in some manner. The common practice is to consider the "bad" values as missing values and then replace each missing value by linear interpolation between the nearest data values on each side of the missing value. (See [7]).

Next, the following parameters must be determined:

A. The largest frequency, f_α, which is present in the data. This is commonly found by visually determining the shortest period in the data.

B. The sampling frequency, f_s, which must be at least $2f_\alpha$.

C. The cut-off frequency, f_c, which is chosen to be at least as great as the highest frequency of interest present in the data.

D. The termination frequency, f_T. This should be chosen such that either, (1) no frequencies present in the data are in the interval (f_c,f_T) or, (2) frequencies appearing in (f_c,f_T) have no significant amplitude.

E. The value of N and hence the number of weights, 2N+1, of the filter.

From the above, the corresponding frequency ratios may be found from $r = \frac{f}{f_s}$. That is, $r_c = \frac{f_c}{f_s}$, $r_T = \frac{f_T}{f_s}$, $r_d = \frac{\Delta f}{f_s}$.

7.2 Empirical error bounds for Martin-Graham filters

Empirical error bounds are found by recovering the digital filter's transfer function, i.e., computing

$$H_N(f_j) = \sum_{n=-N}^{N} h_n e^{2n\pi i\frac{f_j}{f_s}},$$

$j=1,2, \ldots,k$, for various values of the parameters of section 7.1. The recovered values are then compared with the designed or ideal transfer function values at the f_j. An expression for the error ϵ is then determined in terms of N and the other parameters.

The following error bounds were obtained by transfer function recoveries and comparison with bounds obtained by the method of section 4.5.

I. Martin-Graham smoothing filter.

For a maximum error ϵ [see (3.47)] of about .01, take

$$N \geq \frac{1.25}{r_d} = \frac{1.25 f_s}{\Delta f} \tag{7.1}$$

This gives a maximum error of 1% (ϵ referred to unity) between the actual transfer function and the designed transfer function. Note that the error does not change with r_c, r_d held constant.

The bound given by the method of section 4.5 was compared with the results of computation with r_c values ranging from .025 to .2, r_d values ranging from .021 to .11, and N values up to 100. It was found to be about 5 times too large. Hence, in terms of the frequency ratio,

$$\epsilon \doteq \frac{1}{5\pi} \log\frac{4N^2 r_d^{\,2}}{4N^2 r_d^{\,2}-1} \tag{7.2}$$

where "log" denotes the natural logarithm.

II. Martin-Graham first derivative filter.

Comparison of recoveries for $f \geq f_T$, i.e., where $Y^1(f)$ is ideally zero, and the bound obtained by the method of section 4.5 yielded, over the same range of frequencies ratio and N values given above, the expression

$$\epsilon' \doteq \frac{f_s}{4} \left[(r_c+r_T) \log \frac{4N^2 r_d^{\,2}}{4N^2 r_d^{\,2}-1} + \frac{2}{\pi N(4N^2 r_d^{\,2}-1)}\right] . \tag{7.3}$$

III. Martin-Graham second derivative filter.

As above, the following expression was found

$$\epsilon'' \doteq \frac{f_s^{\,2}}{2}\left[\pi(r_c^2+r_T^2) \log \frac{4N^2 r_d^{\,2}}{4N^2 r_d^{\,2}-1} + \frac{r_c+r_T}{N(4N^2 r_d^{\,2}-1)}\right]. \tag{7.4}$$

IV. Martin-Graham band-pass filters.

The error can be as much as the sum of the errors in the low-pass filters from which the band-pass filter is derived (see section 4.7). Hence, in band-pass smoothing the error may be twice that obtained with a low-pass smoothing filter having the same roll-off length Δf.

The values of e' given by (7.3) become too large for small r_d, but are still useable for r_d=.021. The values of ϵ'' given by (7.4) are too small for large r_d and small r_c. The actual value may be as much as $\frac{4}{3}\,\epsilon''$ for r_d values from .07 to .11 and r_c values of .025 to .07. However, it is still useable. ϵ' and ϵ'' are values for the error on the rejection band $|f| \geq f_T$ ($|r| \geq r_T$). The error on the pass-band $|f| \leq f_c$ ($|r| \leq r_c$) is essentially the same. For the first derivative filter, the amplitude at f_c ideally is $2\pi f_c = 2\pi f_s r_c$. For an error of 1% of $2\pi f_c$, we need

$$\epsilon' = .01(2\pi f_s r_c)$$

$$= (.02)\pi r_c f_s$$

$$= (.08)(\pi r_c)\frac{f_s}{4} .$$

Comparing with (7.3), we see that N must be taken such that

$$(.08)\pi r_c \doteq (r_c + r_T) \log \frac{4N^2 r_d^{\,2}}{4N^2 r_d^{\,2} - 1} + \frac{2}{\pi N(4N^2 r_d^{\,2} - 1)} . \qquad (7.5)$$

For the second derivative, the amplitude at f_c ideally is $4\pi^2 f_c^{\,2} = 4\pi^2 f_s^{\,2} r_c^{\,2}$. Similar to the above, we find that for an error of 1% of $4\pi^2 f_c^{\,2}$, we need to take N such that

$$(.08)\pi^2 r_c^{\,2} \doteq \pi(r_c^{\,2} + r_T^{\,2}) \log \frac{4N^2 r_d^{\,2}}{4N^2 r_d^{\,2} - 1} + \frac{r_c + r_T}{N(4N^2 r_d^{\,2} - 1)} . \qquad (7.6)$$

7.3 Programs and examples for Martin-Graham filters

When the appropriate filter has been chosen, the data edited, and the parameters of section 7.1 determined, the filtering can be performed. The weights, h_n, of the filter are computed from the weight function h(t) and (3.44). If the data has a polynomial content, constrained weights $\bar{h}_n$ are computed from the h_n (see section 4.6, section 5.2, and appendix A). Then the output is computed using (3.45).

Consider the following examples. We take as the input the function

$$g(t) = A_1 \cos 2\pi f_1 t + A_2 \sin 2\pi f_2 t + A_3 \cos 2\pi f_3 t + A_4 . \qquad (7.7)$$

Using the Martin-Graham filters, we perform the operations of smoothing; smoothing and finding the first derivative; and smoothing and finding the second derivative. The sampled version of g(t) is

$$g_n = g\left(\frac{n}{f_s}\right) = A_1 \cos 2\pi\frac{n}{f_s} f_1 + A_2 \sin 2\pi\frac{n}{f_s} f_2 + A_3 \cos 2\pi\frac{n}{f_s} f_3 + A_4, \tag{7.8}$$

and going to the frequency ratio, $r = \frac{f}{f_s}$, gives

$$g_n = A_1 \cos 2\pi n r_1 + A_2 \sin 2\pi n r_2 + A_3 \cos 2\pi n r_3 + A_4. \tag{7.9}$$

The following PDQ FORTRAN program was run on the IBM 1620. It is sectioned by comment cards which state what each part of the program does. Table 7.1 gives the frequencies used for the various runs. Table 7.2 gives the frequency ratios, coefficients of the input and desired outputs, the corresponding program symbols, and the values of these parameters for each run. The value of N used, and hence the number of weights for each run, is given by the last two digits in the run number. That is, Run 2.20 reads, "Run 2 with N = 20".

The following symbolism was selected:

FS: The sampling frequency, f_s.

HO: The central smoothing weight, h_o.

DDHO: The central smoothing and second derivative weight.

H(I): The smoothing weights h_I, $I \neq 0$.

DH(I): The smoothing and first derivative weights.

DDH(I): The smoothing and second derivative weights, $I \neq 0$.

TF1: The recovered transfer function for smoothing.

TF2: The recovered transfer function for smoothing and the first derivative divided by 2π.

TF3: The recovered transfer function for smoothing and the second derivative divided by $4\pi^2$.

Z(I): The input samples g_n, n = I-N-1.

Z1(I): The input samples on the range of interest.

Z2(I): The desired smoothed output.

DZ(I): The desired smoothed and first derivative output.

DDZ(I): The desired smoothed and second derivative output.

S1: The actual smoothed output.

S2: The actual smoothed and first derivative output.

S3: The actual smoothed and second derivative output.

The following weight properties are used in the program:

1) Smoothing: $h_{-n} = h_n$,

2) Smoothing and first derivative: $y^1_{-n} = -y^1_n$

3) Smoothing and second derivative: $y^2_{-n} = y^2_n$.

4) Smoothing and integrating: $y^{(-1)}_{-n} = -y^{(-1)}_n$

The results for each run follow the program.

In Run 2.20, the error ϵ in the transfer function is about .011. At t = .8, the actual output differs from the desired output by .037. Referred to the desired output, this is an error of about 2 %.

```
C   C  WEIGHTS, RECOVERY, FILTERING OF DATA, PDQ FORTRAN
       DIMENSION A(30), H(30), DH(30), DDH(30), Z(101)
       DIMENSION Z1(40), Z2(40), DDZ(40), DZ(40)
       P=3.14159
       READ 1,XN1,TC,TD
     1 FORMAT (3F10.0)
       READ 5,T1,T2,T3,FS
     5 FORMAT (4F10.0)
       READ 14,A1,A2,A3,A4
    14 FORMAT (4F10.0)
       READ 18,B1,B2,B3,B4
    18 FORMAT (4F10.0)
       N1=XN1
C   C  UNCONSTRAINED WEIGHTS
       HO=2.*TC+TD
       RT=TC+TD
       DDHO=TD*TD*(RT+TC)/(P*P*P)-(RT*RT*RT+TC*TC*TC)/(6.*P)
       DDHO=8.*P*P*P*DDHO
       DO 2 I=1,N1
       X=I
       A(I)=SINF(HO*X*P)/(X*P)
       H(I)=A(I)*COSF(TD*X*P)/(1.-4.*TD*TD*X*X)
       DH(I)=RT*COSF(RT*X*2.*P)+TC*COSF(TC*X*2.*P)
       DH(I)=DH(I)-H(I)*(1.-12.*TD*TD*X*X)
       DH(I)=DH(I)/(X*(1.-4.*TD*TD*X*X))
       DDH(I)=-2.*DH(I)*(1.-12.*TD*TD*X*X)+24.*TD*TD*X*H(I)
       DDH(I)=DDH(I)-2.*P*TC*TC*SINF(2.*TC*P*X)
       DDH(I)=DDH(I)-2.*P*RT*RT*SINF(2.*RT*P*X)
     2 DDH(I)=DDH(I)/(X*(1.-4.*TD*TD*X*X))
       N2=N1
C   C  CONSTRAINED SMOOTHING WEIGHTS
       S1=0
       DO 3 I=1,N2
     3 S1=S1+H(I)
       FN1=2*N1+1
       S1=1.-(HO+2.*S1)
       S1=S1/FN1
       DO 4 I=1,N1
     4 H(I)=H(I)+S1
       HO=HO+S1
C   C  TRANSFER FUNCTION RECOVERY
       Z1=100.*TC-1.
       Z2=Z1+6.
       Z3=Z2+100.*TD
       Z4=Z3+6.
       DO 13 K=1,57
       TF1=0
       TF2=0
       TF3=0
       X=K
       IF(X-Z1) 6,66,66
     6 Y=K-1
       Y=.01*Y
       GO TO 10
    66 IF(X-Z2) 7,77,77
     7 Y=Y+.005
       GO TO 10
    77 IF(X-Z3) 8,88,88
     8 Y=Y+.01
       GO TO 10
```

```
   88 IF(X-Z4) 9,99,99
    9 Y=Y+.005
      GO TO 10
   99 Y=Y+.01
   10 CONTINUE
      DO 11 I=1,N2
      X=I
      TF1=TF1+2.*H(I)*COSF(2.*X*P*Y)
      TF2=TF2+2.*DH(I)*SINF(2.*X*P*Y)
   11 TF3=TF3+2.*DDH(I)*COSF(2.*X*P*Y)
      TF1=HO+TF1
      TF2=-FS*TF2/(2.*P)
      TF3=DDHO+TF3
      TF3=FS*FS*TF3/(4.*P*P)
      Y1=Y*FS
      PUNCH 12,Y1,TF1,TF2,TF3
   12 FORMAT (E7.3, 3E15.8)
   13 CONTINUE
C  C  INPUT DATA
      P2=P*P
      M1=N1+1
      M=2*N1+40
      DO 15 I=1,M
      T=I-M1
      C1=COSF(2.*P*T1*T)
      S=SINF(2.*P*T2*T)
      C3=COSF(2.*P*T3*T)
   15 Z(I)=A1*C1+A2*S+A3*C3+A4
C  C  INPUT DATA ON THE RANGE OF INTEREST
      DO 17 I=1,40
      T=I-1
      C1=COSF(2.*P*T1*T)
      S=SINF(2.*P*T2*T)
      C3=COSF(2.*P*T3*T)
      Z1(I)=A1*C1+A2*S+A3*C3+A4
      Y=T/FS
      PUNCH 16,Y,Z1(I)
   16 FORMAT (E7.3, E15.8)
   17 CONTINUE
C  C  DESIRED OUTPUTS
      DO 20 I=1,40
      T=I-1
      C1=COSF(2.*P*T1*T)
      S=SINF(2.*P*T2*T)
      C3=COSF(2.*P*T3*T)
      Z2(I)=B1*C1+B2*S+B3*C3+B4
      DDZ(I)=-4.*P2*FS*FS*(B1*T1*T1*C1+B2*T2*T2*S+B3*T3*T3*C3)
      C1=SINF(2.*P*T1*T)
      S=COSF(2.*P*T2*T)
      C3=SINF(2.*P*T3*T)
      DZ(I)=-2.*P*FS*(B1*T1*C1-B2*T2*S+B3*T3*C3)
      Y=T/FS
      PUNCH 19,Y,Z2(I),DZ(I),DDZ(I)
   19 FORMAT (E7.3, 3E15.8)
   20 CONTINUE
C  C  ACTUAL OUTPUTS
      DO 23 K=1,40
      M2=K-1
      M3=N1+1
      S1=0
```

```
      S2=0
      S3=0
      T=M2
      T=T/FS
      DO 21 I=1,N1
      K1=M3-I
      K2=I+M2
      K3=M3+I+M2
      S1=S1+H(K1)*Z(K2)+H(I)*Z(K3)
      S2=S2-DH(K1)*Z(K2)+DH(I)*Z(K3)
   21 S3=S3+DDH(K1)*Z(K2)+DDH(I)*Z(K3)
      K4=M3+M2
      S1=HO*Z(K4)+S1
      S2=-FS*S2
      S3=DDHO*Z(K4)+S3
      S3=FS*FS*S3
      PUNCH 22,T,S1,S2,S3
   22 FORMAT (E7.3, 3E15.8)
   23 CONTINUE
      STOP
      END
```

TABLE 7.1
FREQUENCIES

FREQ.	RUN 1.20	RUN 2.20	RUN 3.20	RUN 4.30
f_1	.5	.5	.5	.5
f_2	.9	.95	2.0	2.0
f_3	2.0	1.9	4.0	4.0
f_s	10.0	10.0	10.0	10.0
f_c	1.0	1.0	2.0	2.0
Δf	.6	.6	.6	.6
f_T	1.6	1.6	2.6	2.6

TABLE 7.2
PROGRAM SYMBOLS AND PARAMETER VALUES

FREQ. RATIO	PROGRAM SYMBOLS	PARAMETER VALUES RUN 1.20	RUN 2.20	RUN 3.20	RUN 4.30
r_c	TC	.1	.1	.2	.2
r_T	RT	.16	.16	.26	.26
r_d	TD	.06	.06	.06	.06
r_1	T1	.05	.05	.05	.05
r_2	T2	.09	.095	.2	.2
r_3	T3	.2	.19	.4	.4
COEFFICIENTS*					
A_1	A1	1.0	1.0	1.0	1.0
A_2	A2	1.0	2.0	2.0	2.0
A_3	A3	.5	1.5	1.5	1.5
A_4	A4	.5	1.0	1.0	1.0
	B1	1.0	1.0	1.0	1.0
	B2	1.0	2.0	2.0	2.0
	B3	0.0	0.0	0.0	0.0
	B4	.5	1.0	1.0	1.0

*The coefficients B1, B2, B3, and B4 are for the desired output.

Run 1.20. Recovered transfer functions.

f	TF1	TF2	TF3
.000E-50	.10000000E 01	-.00000000E-50	-.13118759E-02
.100E 00	.99797082E 00	.10644201E 00	-.10946376E-01
.200E 00	.99514616E 00	.20274402E 00	-.39351767E-01
.300E 00	.99689773E 00	.29448908E 00	-.87443287E-01
.400E 00	.10028117E 01	.39483205E 00	-.15869925E 00
.500E 00	.10056242E 01	.50387711E 00	-.25324265E 00
.600E 00	.10002252E 01	.60704498E 00	-.36411447E 00
.700E 00	.99285677E 00	.69780276E 00	-.48710996E 00
.750E 00	.99266489E 00	.74289948E 00	-.55597912E 00
.800E 00	.99624218E 00	.79162209E 00	-.63277556E 00
.850E 00	.10028210E 01	.84539667E 00	-.71897448E 00
.900E 00	.10095384E 01	.90275637E 00	-.81362836E 00
.950E 00	.10114761E 01	.95858893E 00	-.91209207E 00
.100E 01	.10023041E 01	.10044468E 01	-.10055591E 01
.110E 01	.92563225E 00	.10252742E 01	-.11271631E 01
.120E 01	.75029065E 00	.90216597E 00	-.10811058E 01
.130E 01	.50077482E 00	.64488536E 00	-.83823027E 00
.140E 01	.25002918E 00	.34476281E 00	-.48412609E 00
.150E 01	.72924460E-01	.11260134E 00	-.16974264E 00
.160E 01	-.32426400E-02	.19521540E-02	-.20212799E-02
.165E 01	-.11377890E-01	-.13822814E-01	.24411391E-01
.170E 01	-.84137400E-02	-.13544579E-01	.24458559E-01
.175E 01	-.11392600E-02	-.57593521E-02	.10723431E-01
.180E 01	.52603800E-02	.26600947E-02	-.52313811E-02
.185E 01	.79699400E-02	.76060798E-02	-.15449837E-01
.190E 01	.65757700E-02	.78909192E-02	-.16755633E-01
.200E 01	-.21774900E-02	-.37392119E-04	-.40204820E-03
.210E 01	-.58878800E-02	-.52531985E-02	.12614394E-01
.220E 01	-.58409000E-03	-.16588389E-02	.49615192E-02
.230E 01	.46315200E-02	.32130305E-02	-.83732144E-02
.240E 01	.26071700E-02	.24427648E-02	-.77437910E-02
.250E 01	-.27621400E-02	-.14786985E-02	.37839675E-02
.260E 01	-.36313500E-02	-.24903026E-02	.85511753E-02
.270E 01	.64339000E-03	.12064591E-03	.70627570E-03
.280E 01	.35876200E-02	.20251410E-02	-.73173091E-02
.290E 01	.12612300E-02	.77131372E-03	-.42105583E-02
.300E 01	-.26257000E-02	-.13044897E-02	.46011116E-02
.310E 01	-.25406600E-02	-.11997365E-02	.61619848E-02
.320E 01	.10930100E-02	.54057674E-03	-.11957794E-02
.330E 01	.29542400E-02	.12258077E-02	-.64062478E-02
.340E 01	.55387000E-03	.96642464E-04	-.21880954E-02
.350E 01	-.24929200E-02	-.96748445E-03	.49185261E-02
.360E 01	-.18729000E-02	-.49995495E-03	.46417541E-02
.370E 01	.13618800E-02	.58565360E-03	-.22487438E-02
.380E 01	.25360100E-02	.66474301E-03	-.55934707E-02
.390E 01	.76130000E-04	-.20482956E-03	-.74151273E-03
.400E 01	-.24060800E-02	-.63557844E-03	.50422546E-02
.410E 01	-.13945100E-02	-.90762957E-04	.34035565E-02
.420E 01	.15722800E-02	.48410550E-03	-.31038074E-02
.430E 01	.22277100E-02	.25444567E-03	-.49991779E-02
.440E 01	-.30863000E-03	-.30625590E-03	.33611586E-03
.450E 01	-.23597500E-02	-.31051454E-03	.50543043E-02
.460E 01	-.10061100E-02	.15942691E-03	.23483579E-02
.470E 01	.17721900E-02	.30706919E-03	-.37513775E-02
.480E 01	.19841000E-02	-.58141577E-04	-.43462630E-02
.490E 01	-.65485000E-03	-.28422375E-03	.14058413E-02
.500E 01	-.23433100E-02	-.19021419E-07	.51270327E-02

Run 1.20. Input over the range of interest.

t	Z1(t)
.000E-50	.20000000E 01
.100E 00	.21413919E 01
.200E 00	.18093361E 01
.300E 00	.16753913E 01
.400E 00	.17340388E 01
.500E 00	.13090206E 01
.600E 00	.96808960E-01
.700E 00	-.12212563E 01
.800E 00	-.16958132E 01
.900E 00	-.12263299E 01
.100E 01	-.58778910E 00
.110E 01	-.35933910E 00
.120E 01	-.23177496E 00
.130E 01	.38400309E 00
.140E 01	.13435080E 01
.150E 01	.18090172E 01
.160E 01	.13316611E 01
.170E 01	.49590509E 00
.180E 01	.21995894E 00
.190E 01	.63697291E 00
.200E 01	.10489406E 01
.210E 01	.96814547E 00
.220E 01	.77917473E 00
.230E 01	.11090443E 01
.240E 01	.18078412E 01
.250E 01	.20000066E 01
.260E 01	.11898459E 01
.270E 01	-.66488190E-01
.280E 01	-.83885020E 00
.290E 01	-.93397380E 00
.300E 01	-.95105200E 00
.310E 01	-.12651218E 01
.320E 01	-.13980790E 01
.330E 01	-.67970750E 00
.340E 01	.71357536E 00
.350E 01	.18089978E 01
.360E 01	.19615601E 01
.370E 01	.15595957E 01
.380E 01	.13862599E 01
.390E 01	.15427700E 01

Run 1.20. Desired outputs.

t	Smoothed Z2(t)	1st Derivative DZ(t)	2nd Derivative DDZ(t)
.000E-50	.15000000E 01	.56548620E 01	-.98695878E 01
.100E 00	.19868829E 01	.38037554E 01	-.26520906E 02
.200E 00	.22138439E 01	.56114911E 00	-.36918729E 02
.300E 00	.20799007E 01	-.32503322E 01	-.37526523E 02
.400E 00	.15795324E 01	-.65923646E 01	-.27688979E 02
.500E 00	.80902060E 00	-.85196792E 01	-.98816658E 01
.600E 00	-.57702570E-01	-.84650394E 01	.11002238E 02
.700E 00	-.81675000E 00	-.64126361E 01	.29111678E 02
.800E 00	-.12913022E 01	-.29062224E 01	.39395685E 02
.900E 00	-.13808338E 01	.11108591E 01	.39118474E 02
.100E 01	-.10877891E 01	.45748552E 01	.28665593E 02
.110E 01	-.51385310E 00	.66144971E 01	.11394594E 02
.120E 01	.17272979E 00	.68019809E 01	-.74204176E 01
.130E 01	.78851564E 00	.52658747E 01	-.22220744E 02
.140E 01	.11890066E 01	.26327922E 01	-.28864472E 02
.150E 01	.13090172E 01	-.18222171E 00	-.25870408E 02
.160E 01	.11771446E 01	-.22699080E 01	-.14821747E 02
.170E 01	.90040828E 00	-.30130995E 01	.19056161E 00
.180E 01	.62447306E 00	-.22756563E 01	.13905253E 02
.190E 01	.48247401E 00	-.43553584E 00	.21586242E 02
.200E 01	.54894060E 00	.17474137E 01	.20542883E 02
.210E 01	.81362636E 00	.33863211E 01	.10996902E 02
.220E 01	.11836763E 01	.37636986E 01	-.39765130E 01
.230E 01	.15135599E 01	.25751092E 01	-.19416283E 02
.240E 01	.16533449E 01	.42256899E-01	-.30049203E 02
.250E 01	.15000066E 01	-.31415225E 01	-.31977530E 02
.260E 01	.10353242E 01	-.60178027E 01	-.23949877E 02
.270E 01	.33801188E 00	-.76582527E 01	-.78145864E 01
.280E 01	-.43433296E 00	-.74568799E 01	.11992048E 02
.290E 01	-.10884676E 01	-.53280230E 01	.29769379E 02
.300E 01	-.14510520E 01	-.17475506E 01	.40281818E 02
.310E 01	-.14196459E 01	.23770048E 01	.40359512E 02
.320E 01	-.99358050E 00	.59687149E 01	.29875160E 02
.330E 01	-.27518875E 00	.80962657E 01	.11793730E 02
.340E 01	.55908404E 00	.82456119E 01	-.87212573E 01
.350E 01	.13089978E 01	.64655110E 01	-.25869906E 02
.360E 01	.18070335E 01	.33430089E 01	-.34964109E 02
.370E 01	.19640926E 01	-.18254309E 00	-.33823464E 02
.380E 01	.17907802E 01	-.31087390E 01	-.23390364E 02
.390E 01	.13882812E 01	-.46728742E 01	-.73792113E 01

Run 1.20. Actual outputs.

t	Smoothed S1	1st Derivative S2	2nd Derivative S3
.000E-50	.15045354E 01	.56721806E 01	-.10031434E 02
.100E 00	.19970064E 01	.38109615E 01	-.26747757E 02
.200E 00	.22279052E 01	.55427308E 00	-.37171376E 02
.300E 00	.20935504E 01	-.32722799E 01	-.37763351E 02
.400E 00	.15882834E 01	-.66266840E 01	-.27867256E 02
.500E 00	.81087941E 00	-.85605104E 01	-.99597611E 01
.600E 00	-.62149062E-01	-.85048700E 01	.11049072E 02
.700E 00	-.82612818E 00	-.64441304E 01	.29271869E 02
.800E 00	-.13043409E 01	-.29238554E 01	.39620482E 02
.900E 00	-.13953876E 01	.11095947E 01	.39345057E 02
.100E 01	-.11001086E 01	.45888662E 01	.28843974E 02
.110E 01	-.52013754E 00	.66394216E 01	.11496990E 02
.120E 01	.17365562E 00	.68315450E 01	-.74053338E 01
.130E 01	.79444918E 00	.52938569E 01	-.22290500E 02
.140E 01	.11964518E 01	.26547608E 01	-.28996224E 02
.150E 01	.13156452E 01	-.16804116E 00	-.26020127E 02
.160E 01	.11820574E 01	-.22627312E 01	-.14942387E 02
.170E 01	.90280760E 00	-.30103346E 01	.12268360E 00
.180E 01	.62337475E 00	-.22740332E 01	.13880260E 02
.190E 01	.47824776E 00	-.43242794E 00	.21574888E 02
.200E 01	.54440436E 00	.17527655E 01	.20517271E 02
.210E 01	.81255887E 00	.33922501E 01	.10938109E 02
.220E 01	.11879120E 01	.37666320E 01	-.40816116E 01
.230E 01	.15218073E 01	.25710047E 01	-.19571985E 02
.240E 01	.16628000E 01	.28258741E-01	-.30238055E 02
.250E 01	.15084564E 01	-.31658828E 01	-.32154616E 02
.260E 01	.10413034E 01	-.60501405E 01	-.24059611E 02
.270E 01	.33964794E 00	-.76935647E 01	-.78197993E 01
.280E 01	-.43919722E 00	-.74884514E 01	.12094081E 02
.290E 01	-.11002328E 01	-.53490081E 01	.29954085E 02
.300E 01	-.14668365E 01	-.17529024E 01	.40512243E 02
.310E 01	-.14345700E 01	.23889526E 01	.40591663E 02
.320E 01	-.10037791E 01	.59957294E 01	.30057299E 02
.330E 01	-.27940151E 00	.81329167E 01	.11876349E 02
.340E 01	.56052073E 00	.82847717E 01	-.87627748E 01
.350E 01	.13156258E 01	.65000505E 01	-.26019624E 02
.360E 01	.18179545E 01	.33673742E 01	-.35174977E 02
.370E 01	.19766374E 01	-.17111221E 00	-.34043700E 02
.380E 01	.18008065E 01	-.31096643E 01	-.23582420E 02
.390E 01	.13926950E 01	-.46827420E 01	-.75203155E 01

See figures 7.1 and 7.2, pages 114 and 115.

Run 2.20. Input on the range of interest.

t	Z1(t)
.000E-50	.35000000E 01
.100E 00	.36274107E 01
.200E 00	.25751186E 01
.300E 00	.21823776E 01
.400E 00	.27722949E 01
.500E 00	.27394576E 01
.600E 00	.79557440E 00
.700E 00	-.20318853E 01
.800E 00	-.32932433E 01
.900E 00	-.19044190E 01
.100E 01	.59547310E 00
.110E 01	.18734152E 01
.120E 01	.14509463E 01
.130E 01	.92990740E 00
.140E 01	.16398419E 01
.150E 01	.27896500E 01
.160E 01	.25112433E 01
.170E 01	.45319630E 00
.180E 01	-.14425950E 01
.190E 01	-.10865011E 01
.200E 01	.12879099E 01
.210E 01	.33852534E 01
.220E 01	.35193545E 01
.230E 01	.23964943E 01
.240E 01	.18789242E 01
.250E 01	.24141995E 01
.260E 01	.24604273E 01
.270E 01	.64480020E 00
.280E 01	-.21362846E 01
.290E 01	-.34471109E 01
.300E 01	-.20816204E 01
.310E 01	.52717530E 00
.320E 01	.20028098E 01
.330E 01	.17244565E 01
.340E 01	.12223375E 01
.350E 01	.19002987E 01
.360E 01	.30762402E 01
.370E 01	.28730387E 01
.380E 01	.81532020E 00
.390E 01	-.12359836E 01

The transfer functions for Run 2.20 are the same as for Run 1.20. The frequencies f_2 and f_3 were chosen such that they appear where the transfer function has a relatively large error for each in smoothing.

Run 2.20. Desired outputs.

t	Smoothed Z2(t)	1st Derivative DZ(t)	2nd Derivative DDZ(t)
.000E-50	.20000000E 01	.11938042E 02	-.98695878E 01
.100E 00	.30752225E 01	.89029224E 01	-.49439679E 02
.200E 00	.36685695E 01	.25481186E 01	-.74239047E 02
.300E 00	.35396201E 01	-.51457837E 01	-.75343517E 02
.400E 00	.26781152E 01	-.11690270E 02	-.51829733E 02
.500E 00	.13128752E 01	-.14932650E 02	-.11147463E 02
.600E 00	-.16056850E 00	-.13789710E 02	.33390021E 02
.700E 00	-.13092641E 01	-.86185972E 01	.67136173E 02
.800E 00	-.18050697E 01	-.10970380E 01	.79102480E 02
.900E 00	-.15313713E 01	.63460558E 01	.65691925E 02
.100E 01	-.61804350E 00	.11353726E 02	.31889993E 02
.110E 01	.60691420E 00	.12434841E 02	-.10493541E 02
.120E 01	.17319999E 01	.94562217E 01	-.46920600E 02
.130E 01	.24033347E 01	.36651402E 01	-.65140906E 02
.140E 01	.24435997E 01	-.27632956E 01	-.59394571E 02
.150E 01	.19079905E 01	-.74952425E 01	-.32351089E 02
.160E 01	.10583625E 01	-.88560850E 01	.58806474E 01
.170E 01	.26517030E 00	-.64133147E 01	.41322436E 02
.180E 01	-.12814780E 00	-.11223806E 01	.61034916E 02
.190E 01	.69286900E-01	.50145807E 01	.57659391E 02
.200E 01	.82441320E 00	.96580243E 01	.32015644E 02
.210E 01	.18882153E 01	.10961359E 02	-.71475097E 01
.220E 01	.28806549E 01	.82331297E 01	-.46166189E 02
.230E 01	.34232900E 01	.21997103E 01	-.71198657E 02
.240E 01	.32736022E 01	-.52246427E 01	-.73046335E 02
.250E 01	.24142373E 01	-.11582958E 02	-.50387988E 02
.260E 01	.10657772E 01	-.14714394E 02	-.10303582E 02
.270E 01	-.38205070E 00	-.13497869E 02	.34100402E 02
.280E 01	-.14976535E 01	-.82434618E 01	.68149563E 02
.290E 01	-.19500681E 01	-.59601818E 00	.80609807E 02
.300E 01	-.16180518E 01	.70168343E 01	.67519496E 02
.310E 01	-.62856450E 00	.12202990E 02	.33525547E 02
.320E 01	.68832620E 00	.13409593E 02	-.97354072E 01
.330E 01	.19124074E 01	.10436504E 02	-.47649667E 02
.340E 01	.26751995E 01	.44842623E 01	-.67646423E 02
.350E 01	.27820194E 01	-.22779851E 01	-.63492183E 02
.360E 01	.22725463E 01	-.74734449E 01	-.37379897E 02
.370E 01	.13995972E 01	-.93434638E 01	.90358681E 00
.380E 01	.53419230E 00	-.73519577E 01	.37436183E 02
.390E 01	.30476800E-01	-.23599962E 01	.59042124E 02

Run 2.20. Actual outputs.

t	Smoothed S1	1st Derivative S2	2nd Derivative S3
.000E-50	.20154878E 01	.12045973E 02	-.11041620E 02
.100E 00	.30971031E 01	.89155165E 01	-.50404191E 02
.200E 00	.36872694E 01	.25226223E 01	-.74375279E 02
.300E 00	.35564000E 01	-.51573711E 01	-.75311883E 02
.400E 00	.26961842E 01	-.11717893E 02	-.52501832E 02
.500E 00	.13258467E 01	-.15040630E 02	-.12261396E 02
.600E 00	-.16579159E 00	-.13967839E 02	.33067783E 02
.700E 00	-.13370774E 01	-.87584182E 01	.68289526E 02
.800E 00	-.18423126E 01	-.10952590E 01	.80894529E 02
.900E 00	-.15573091E 01	.64767132E 01	.66607086E 02
.100E 01	-.62278052E 00	.11500090E 02	.31397538E 02
.110E 01	.61629707E 00	.12506166E 02	-.11472647E 02
.120E 01	.17432870E 01	.94662856E 01	-.47266475E 02
.130E 01	.24131902E 01	.36810689E 01	-.64896809E 02
.140E 01	.24566893E 01	-.27293315E 01	-.59538814E 02
.150E 01	.19242084E 01	-.75068831E 01	-.33329921E 02
.160E 01	.10667777E 01	-.89584918E 01	.49231610E 01
.170E 01	.25453440E 00	-.65483543E 01	.41571858E 02
.180E 01	-.15447227E 00	-.11707356E 01	.62482610E 02
.190E 01	.45440250E-01	.51060713E 01	.58962975E 02
.200E 01	.81959411E 00	.98160733E 01	.31974414E 02
.210E 01	.19026873E 01	.11066381E 02	-.82875370E 01
.220E 01	.29017031E 01	.82426470E 01	-.47149863E 02
.230E 01	.34409083E 01	.21686487E 01	-.71341566E 02
.240E 01	.32887141E 01	-.52406639E 01	-.72959075E 02
.250E 01	.24304671E 01	-.11609264E 02	-.50975316E 02
.260E 01	.10775115E 01	-.14816202E 02	-.11380296E 02
.270E 01	-.38771926E 00	-.13670845E 02	.33745373E 02
.280E 01	-.15257821E 01	-.83829090E 01	.69263233E 02
.290E 01	-.19882024E 01	-.59549004E 00	.82427016E 02
.300E 01	-.16452934E 01	.71510027E 01	.68515082E 02
.310E 01	-.63408873E 00	.12359475E 02	.33087545E 02
.320E 01	.69812756E 00	.13492625E 02	-.10741499E 02
.330E 01	.19250822E 01	.10453800E 02	-.48069999E 02
.340E 01	.26866783E 01	.45024557E 01	-.67448987E 02
.350E 01	.27966715E 01	-.22424527E 01	-.63635556E 02
.360E 01	.22906269E 01	-.74820601E 01	-.38367769E 02
.370E 01	.14104324E 01	-.94451399E 01	-.12686000E 00
.380E 01	.52596078E 00	-.74938559E 01	.37577600E 02
.390E 01	.54566600E-02	-.24224347E 01	.60433601E 02

Run 3.20. Recovered transfer functions.

f	TF1	TF2	TF3
.000E-50	.10000000E 01	-.00000000E-50	.86044627E-03
.100E 00	.99945238E 00	.10761383E 00	-.10389035E-01
.200E 00	.99863960E 00	.20400454E 00	-.41397524E-01
.300E 00	.99897643E 00	.29440003E 00	-.89287873E-01
.400E 00	.10005081E 01	.39296784E 00	-.15734707E 00
.500E 00	.10015378E 01	.50204665E 00	-.25000005E 00
.600E 00	.10006253E 01	.60835788E 00	-.36397262E 00
.700E 00	.99870627E 00	.70226832E 00	-.49198182E 00
.800E 00	.99833411E 00	.79240672E 00	-.63544748E 00
.900E 00	.10003624E 01	.89362575E 00	-.80488189E 00
.100E 01	.10022588E 01	.10048168E 01	-.10034423E 01
.110E 01	.10010699E 01	.11093631E 01	-.12186314E 01
.120E 01	.99788668E 00	.11994607E 01	-.14398548E 01
.130E 01	.99730807E 00	.12893453E 01	-.16784461E 01
.140E 01	.10010904E 01	.13955016E 01	-.19546465E 01
.150E 01	.10042013E 01	.15101747E 01	-.22632507E 01
.160E 01	.10007927E 01	.16095504E 01	-.25729319E 01
.170E 01	.99441370E 00	.16912238E 01	-.28758868E 01
.175E 01	.99386400E 00	.17347289E 01	-.30381955E 01
.180E 01	.99666469E 00	.17859937E 01	-.32179358E 01
.185E 01	.10023570E 01	.18460814E 01	-.34178864E 01
.190E 01	.10084103E 01	.19107547E 01	-.36314467E 01
.195E 01	.10101403E 01	.19697331E 01	-.38398498E 01
.200E 01	.10012704E 01	.20077460E 01	-.40125424E 01
.210E 01	.92604993E 00	.19529443E 01	-.40982205E 01
.220E 01	.75147569E 00	.16525654E 01	-.36371321E 01
.230E 01	.50108251E 00	.11436934E 01	-.26347093E 01
.240E 01	.24911331E 00	.59361745E 00	-.14255807E 01
.250E 01	.72100960E-01	.18685445E 00	-.46301103E 00
.260E 01	-.28700700E-02	.61257277E-03	.18476202E-02
.265E 01	-.10526310E-01	-.24190123E-01	.64760126E-01
.270E 01	-.74207000E-02	-.22251017E-01	.57526921E-01
.275E 01	-.38653000E-03	-.83783146E-02	.18153645E-01
.280E 01	.54924300E-02	.58506912E-02	-.21730061E-01
.285E 01	.76056200E-02	.13754990E-01	-.42844089E-01
.290E 01	.57678000E-02	.13636174E-01	-.39896803E-01
.300E 01	-.28776700E-02	-.57320315E-03	.74476768E-02
.310E 01	-.55256200E-02	-.94010325E-02	.32948817E-01
.320E 01	.31000000E-03	-.29603702E-02	.54166142E-02
.330E 01	.48150100E-02	.56742181E-02	-.25297890E-01
.340E 01	.18449600E-02	.45846017E-02	-.14953360E-01
.350E 01	-.33996600E-02	-.22821846E-02	.15631655E-01
.360E 01	-.32646400E-02	-.45695297E-02	.20224603E-01
.370E 01	.14945900E-02	-.39263350E-03	-.48185396E-02
.380E 01	.37360000E-02	.33800911E-02	-.20731159E-01
.390E 01	.49900000E-03	.19979871E-02	-.54942011E-02
.400E 01	-.32386200E-02	-.16358196E-02	.16611028E-01
.410E 01	-.21445300E-02	-.24325171E-02	.13303063E-01
.420E 01	.19569700E-02	-.24102604E-04	-.91247478E-02
.430E 01	.30798600E-02	.19017655E-02	-.17237979E-01
.440E 01	-.25244000E-03	.11191118E-02	.99852194E-04
.450E 01	-.31156400E-02	-.81849770E-03	.16751383E-01
.460E 01	-.14207000E-02	-.14057018E-02	.84910334E-02
.470E 01	.22879500E-02	-.30348199E-03	-.12012786E-01
.480E 01	.26338200E-02	.92689450E-03	-.14552608E-01
.490E 01	-.83840000E-03	.99433503E-03	.43299174E-02
.500E 01	-.30766900E-02	.48763747E-07	.16693073E-01

Run 3.20. Input over the range of interest.

t	Z1(t)
.000E-50	.35000000E 01
.100E 00	.26396454E 01
.200E 00	.34481107E 01
.300E 00	.87575520E 00
.400E 00	-.18066307E 01
.500E 00	.24999908E 01
.600E 00	.13795795E 01
.700E 00	.20513030E 01
.800E 00	-.52102270E 00
.900E 00	-.30667170E 01
.100E 01	.14999788E 01
.110E 01	.73754350E 00
.120E 01	.18300614E 01
.130E 01	-.29977140E 00
.140E 01	-.24246944E 01
.150E 01	.24999642E 01
.160E 01	.19976199E 01
.170E 01	.32268552E 01
.180E 01	.10970545E 01
.190E 01	-.11646317E 01
.200E 01	.34999576E 01
.210E 01	.26396713E 01
.220E 01	.34480875E 01
.230E 01	.87585440E 00
.240E 01	-.18066762E 01
.250E 01	.24999536E 01
.260E 01	.13796088E 01
.270E 01	.20512811E 01
.280E 01	-.52092470E 00
.290E 01	-.30667660E 01
.300E 01	.14999363E 01
.310E 01	.73756630E 00
.320E 01	.18300321E 01
.330E 01	-.29968060E 00
.340E 01	-.24247501E 01
.350E 01	.24999164E 01
.360E 01	.19976393E 01
.370E 01	.32268247E 01
.380E 01	.10971464E 01
.390E 01	-.11646838E 01

Run 3.20. Desired outputs.

t	Smoothed Z2(t)	1st Derivative DZ(t)	2nd Derivative DDZ(t)
.000E-50	.20000000E 01	.25132720E 02	-.98695878E 01
.100E 00	.38531690E 01	.67956594E 01	-.30975557E 03
.200E 00	.29845913E 01	-.22179345E 02	-.19362355E 03
.300E 00	.41222060E 00	-.22874442E 02	.17983632E 03
.400E 00	-.59309770E 00	.47785073E 01	.29731968E 03
.500E 00	.99999080E 00	.21991130E 02	.16654732E-02
.600E 00	.25930937E 01	.47787586E 01	-.29731867E 03
.700E 00	.15877988E 01	-.22874291E 02	-.17983906E 03
.800E 00	-.98457240E 00	-.22179509E 02	.19362082E 03
.900E 00	-.18531747E 01	.67953974E 01	.30975660E 03
.100E 01	-.21200000E-04	.25132712E 02	.98729430E 01
.110E 01	.19510484E 01	.87375126E 01	-.29098146E 03
.120E 01	.13665723E 01	-.18486037E 02	-.17765692E 03
.130E 01	-.76333620E 00	-.17791409E 02	.19143604E 03
.140E 01	-.12111427E 01	.10753908E 02	.30342050E 03
.150E 01	.99996420E 00	.28274310E 02	.50716949E-02
.160E 01	.32111154E 01	.10754677E 02	-.30341731E 03
.170E 01	.27633813E 01	-.17790923E 02	-.19144412E 03
.180E 01	.63347450E 00	-.18486488E 02	.17764882E 03
.190E 01	.48929400E-01	.87367750E 01	.29098460E 03
.200E 01	.19999576E 01	.25132736E 02	-.98628788E 01
.210E 01	.38531574E 01	.67961828E 01	-.30975352E 03
.220E 01	.29846287E 01	-.22179017E 02	-.19362901E 03
.230E 01	.41225920E 00	-.22874747E 02	.17983085E 03
.240E 01	-.59310580E 00	.47780053E 01	.29732171E 03
.250E 01	.99995360E 00	.21991130E 02	.83225003E-02
.260E 01	.25930856E 01	.47792613E 01	-.29731664E 03
.270E 01	.15878375E 01	-.22873987E 02	-.17984453E 03
.280E 01	-.98453500E 00	-.22179836E 02	.19361536E 03
.290E 01	-.18531862E 01	.67948740E 01	.30975866E 03
.300E 01	-.63700000E-04	.25132695E 02	.98796524E 01
.310E 01	.19510337E 01	.87380040E 01	-.29097937E 03
.320E 01	.13666036E 01	-.18485737E 02	-.17766231E 03
.330E 01	-.76330610E 00	-.17791732E 02	.19143065E 03
.340E 01	-.12111609E 01	.10753395E 02	.30342262E 03
.350E 01	.99991640E 00	.28274310E 02	.11833271E-01
.360E 01	.32110973E 01	.10755190E 02	-.30341519E 03
.370E 01	.27634114E 01	-.17790600E 02	-.19144950E 03
.380E 01	.63350580E 00	-.18486788E 02	.17764342E 03
.390E 01	.48914700E-01	.87362830E 01	.29098669E 03

Run 3.20. Actual outputs.

t	Smoothed S1	1^{st} Derivative S2	2^{nd} Derivative S3
.000E-50	.19966798E 01	.25230053E 02	-.88519600E 01
.100E 00	.38609774E 01	.68308277E 01	-.31145924E 03
.200E 00	.29858278E 01	-.22280317E 02	-.19386771E 03
.300E 00	.41012973E 00	-.22948935E 02	.18075636E 03
.400E 00	-.59110874E 00	.47872926E 01	.29749969E 03
.500E 00	.99513290E 00	.22075609E 02	.10193050E 01
.600E 00	.25989650E 01	.48056716E 01	-.29902235E 03
.700E 00	.15868873E 01	-.22978105E 02	-.18008323E 03
.800E 00	-.98881099E 00	-.22251155E 02	.19454088E 03
.900E 00	-.18531235E 01	.68124424E 01	.30993660E 03
.100E 01	-.64169700E-02	.25230049E 02	.10890582E 02
.110E 01	.19559322E 01	.87806289E 01	-.29268509E 03
.120E 01	.13653205E 01	-.18571891E 02	-.17790110E 03
.130E 01	-.76723489E 00	-.17845091E 02	.19235609E 03
.140E 01	-.12101041E 01	.10787154E 02	.30360045E 03
.150E 01	.99510620E 00	.28384502E 02	.10227100E 01
.160E 01	.32179365E 01	.10806049E 02	-.30512093E 03
.170E 01	.27642776E 01	-.17873931E 02	-.19168832E 03
.180E 01	.63172394E 00	-.18543018E 02	.17856888E 03
.190E 01	.51905730E-01	.87617643E 01	.29116458E 03
.200E 01	.19966371E 01	.25230066E 02	-.88452300E 01
.210E 01	.38609656E 01	.68313535E 01	-.31145716E 03
.220E 01	.29858655E 01	-.22279989E 02	-.19387323E 03
.230E 01	.41016821E 00	-.22949241E 02	.18075090E 03
.240E 01	-.59111675E 00	.47867862E 01	.29750169E 03
.250E 01	.99509570E 00	.22075606E 02	.10259830E 01
.260E 01	.25989569E 01	.48061764E 01	-.29902032E 03
.270E 01	.15869261E 01	-.22977801E 02	-.18008878E 03
.280E 01	-.98877416E 00	-.22251484E 02	.19453546E 03
.290E 01	-.18531351E 01	.68119199E 01	.30993869E 03
.300E 01	-.64594000E-02	.25230035E 02	.10897302E 02
.310E 01	.19559174E 01	.87811226E 01	-.29268303E 03
.320E 01	.13653520E 01	-.18571596E 02	-.17790656E 03
.330E 01	-.76720516E 00	-.17845418E 02	.19235080E 03
.340E 01	-.12101223E 01	.10786646E 02	.30360260E 03
.350E 01	.99505880E 00	.28384506E 02	.10294110E 01
.360E 01	.32179183E 01	.10806554E 02	-.30511888E 03
.370E 01	.27643073E 01	-.17873613E 02	-.19169365E 03
.380E 01	.63175457E 00	-.18543311E 02	.17856364E 03
.390E 01	.51891750E-01	.87612840E 01	.29116652E 03

Run 4.30. Recovered transfer functions.

f	TF1	TF2	TF3
.000E-50	.10000000E 01	-.00000000E-50	-.58318545E-02
.100E 00	.99907998E 00	.10019297E 00	-.80659397E-02
.200E 00	.99931646E 00	.19971041E 00	-.35478026E-01
.300E 00	.10010677E 01	.30012350E 00	-.95055768E-01
.400E 00	.10000894E 01	.40031356E 00	-.16119351E 00
.500E 00	.99886238E 00	.49948591E 00	-.24395872E 00
.600E 00	.10006425E 01	.60006845E 00	-.36286209E 00
.700E 00	.10007752E 01	.70066934E 00	-.49431150E 00
.800E 00	.99880356E 00	.79929887E 00	-.63393498E 00
.900E 00	.99998585E 00	.89969049E 00	-.80956836E 00
.100E 01	.10013162E 01	.10012110E 01	-.10068540E 01
.110E 01	.99909529E 00	.10993790E 01	-.12056404E 01
.120E 01	.99915694E 00	.11988663E 01	-.14354893E 01
.130E 01	.10016514E 01	.13018188E 01	-.16982835E 01
.140E 01	.99977512E 00	.14000287E 01	-.19592377E 01
.150E 01	.99814679E 00	.14973674E 01	-.22406451E 01
.160E 01	.10017739E 01	.16023815E 01	-.25683347E 01
.170E 01	.10010042E 01	.17018492E 01	-.28955224E 01
.175E 01	.99813180E 00	.17471953E 01	-.30542592E 01
.180E 01	.99640572E 00	.17939404E 01	-.32228329E 01
.185E 01	.99803983E 00	.18463800E 01	-.34119223E 01
.190E 01	.10026755E 01	.19046610E 01	-.36206707E 01
.195E 01	.10060471E 01	.19612778E 01	-.38304844E 01
.200E 01	.10005279E 01	.20008798E 01	-.40068302E 01
.210E 01	.92967818E 00	.19528897E 01	-.40957027E 01
.220E 01	.75327729E 00	.16570165E 01	-.36439244E 01
.230E 01	.50035794E 00	.11503241E 01	-.26522678E 01
.240E 01	.24648207E 00	.59209553E 00	-.14181303E 01
.250E 01	.69765660E-01	.17459921E 00	-.43186968E 00
.260E 01	.88390000E-04	-.47688527E-03	-.49678366E-02
.265E 01	-.53850800E-02	-.14656341E-01	.32960773E-01
.270E 01	-.25282200E-02	-.65433446E-02	.17141369E-01
.275E 01	.14668500E-02	.47777932E-02	-.77990086E-02
.280E 01	.28774900E-02	.86491178E-02	-.17510939E-01
.285E 01	.15368000E-02	.43091425E-02	-.99314431E-02
.290E 01	-.66490000E-03	-.26293922E-02	.35498696E-02
.300E 01	-.12712400E-02	-.39808054E-02	.78814335E-02
.310E 01	.11748000E-02	.45218398E-02	-.69657671E-02
.320E 01	.17509000E-03	.13739491E-03	-.10527795E-02
.330E 01	-.92392000E-03	-.37339274E-02	.55163144E-02
.340E 01	.40365000E-03	.22937934E-02	-.24288248E-02
.350E 01	.45518000E-03	.17028016E-02	-.26208541E-02
.360E 01	-.56294000E-03	-.31189940E-02	.33184010E-02
.370E 01	-.29990000E-04	.52071151E-03	.27863372E-04
.380E 01	.44878000E-03	.24772608E-02	-.26030469E-02
.390E 01	-.22763000E-03	-.21068544E-02	.13943591E-02
.400E 01	-.22481000E-03	-.89525265E-03	.11685898E-02
.410E 01	.29683000E-03	.25695913E-02	-.17309233E-02
.420E 01	.18100000E-04	-.86277327E-03	.52383140E-04
.430E 01	-.23712000E-03	-.18644426E-02	.13160377E-02
.440E 01	.10304000E-03	.20775300E-02	-.64149082E-03
.450E 01	.13462000E-03	.40070746E-03	-.60460985E-03
.460E 01	-.13365000E-03	-.22914146E-02	.76680001E-03
.470E 01	-.51090000E-04	.11534379E-02	.16333002E-03
.480E 01	.11976000E-03	.14658183E-02	-.59599753E-03
.490E 01	.97500000E-05	-.21411032E-02	-.33967984E-04
.500E 01	-.10946000E-03	-.18661849E-06	.44758708E-03

The input and desired output for Run 4.30 are the same as for Run 3.20. The number of weights was increased. Note the improved accuracy in the transfer function and the actual outputs.

Run 4.30. Actual outputs.

t	Smoothed S1	1st Derivative S2	2nd Derivative S3
.000E-50	.19985248E 01	.25143768E 02	-.97921200E 01
.100E 00	.38533630E 01	.68050311E 01	-.31032783E 03
.200E 00	.29841874E 01	-.22194413E 02	-.19395645E 03
.300E 00	.41082716E 00	-.22872747E 02	.18008466E 03
.400E 00	-.59418055E 00	.47800346E 01	.29762004E 03
.500E 00	.99965360E 00	.22005412E 02	-.15936600E 00
.600E 00	.25947223E 01	.47902065E 01	-.29819147E 03
.700E 00	.15889840E 01	-.22888646E 02	-.18050508E 03
.800E 00	-.98437667E 00	-.22178530E 02	.19353604E 03
.900E 00	-.18528241E 01	.67948506E 01	.30975645E 03
.100E 01	.77930000E-03	.25143765E 02	.94734070E 01
.110E 01	.19534076E 01	.87448899E 01	-.29200738E 03
.120E 01	.13680091E 01	-.18504907E 02	-.17837570E 03
.130E 01	-.76339220E 00	-.17794939E 02	.19140399E 03
.140E 01	-.12115225E 01	.10749293E 02	.30357342E 03
.150E 01	.99962710E 00	.28282128E 02	-.15596600E 00
.160E 01	.32120402E 01	.10759978E 02	-.30414267E 03
.170E 01	.27632290E 01	-.17810504E 02	-.19182978E 03
.180E 01	.63182935E 00	-.18489304E 02	.17794993E 03
.190E 01	.47116300E-01	.87342318E 01	.29143807E 03
.200E 01	.19984823E 01	.25143779E 02	-.97853900E 01
.210E 01	.38533517E 01	.68055583E 01	-.31032575E 03
.220E 01	.29842250E 01	-.22194087E 02	-.19396191E 03
.230E 01	.41086573E 00	-.22873052E 02	.18007914E 03
.240E 01	-.59418865E 00	.47795285E 01	.29762208E 03
.250E 01	.99961640E 00	.22005411E 02	-.15269400E 00
.260E 01	.25947143E 01	.47907116E 01	-.29818948E 03
.270E 01	.15890228E 01	-.22888340E 02	-.18051059E 03
.280E 01	-.98433947E 00	-.22178857E 02	.19353060E 03
.290E 01	-.18528357E 01	.67943305E 01	.30975857E 03
.300E 01	.73690000E-03	.25143751E 02	.94801200E 01
.310E 01	.19533929E 01	.87453821E 01	-.29200533E 03
.320E 01	.13680404E 01	-.18504611E 02	-.17838113E 03
.330E 01	-.76336249E 00	-.17795265E 02	.19139871E 03
.340E 01	-.12115406E 01	.10748786E 02	.30357559E 03
.350E 01	.99957970E 00	.28282133E 02	-.14927200E 00
.360E 01	.32120224E 01	.10760484E 02	-.30414064E 03
.370E 01	.27632586E 01	-.17810186E 02	-.19183507E 03
.380E 01	.63186026E 00	-.18489595E 02	.17794463E 03
.390E 01	.47102350E-01	.87337504E 01	.29144001E 03

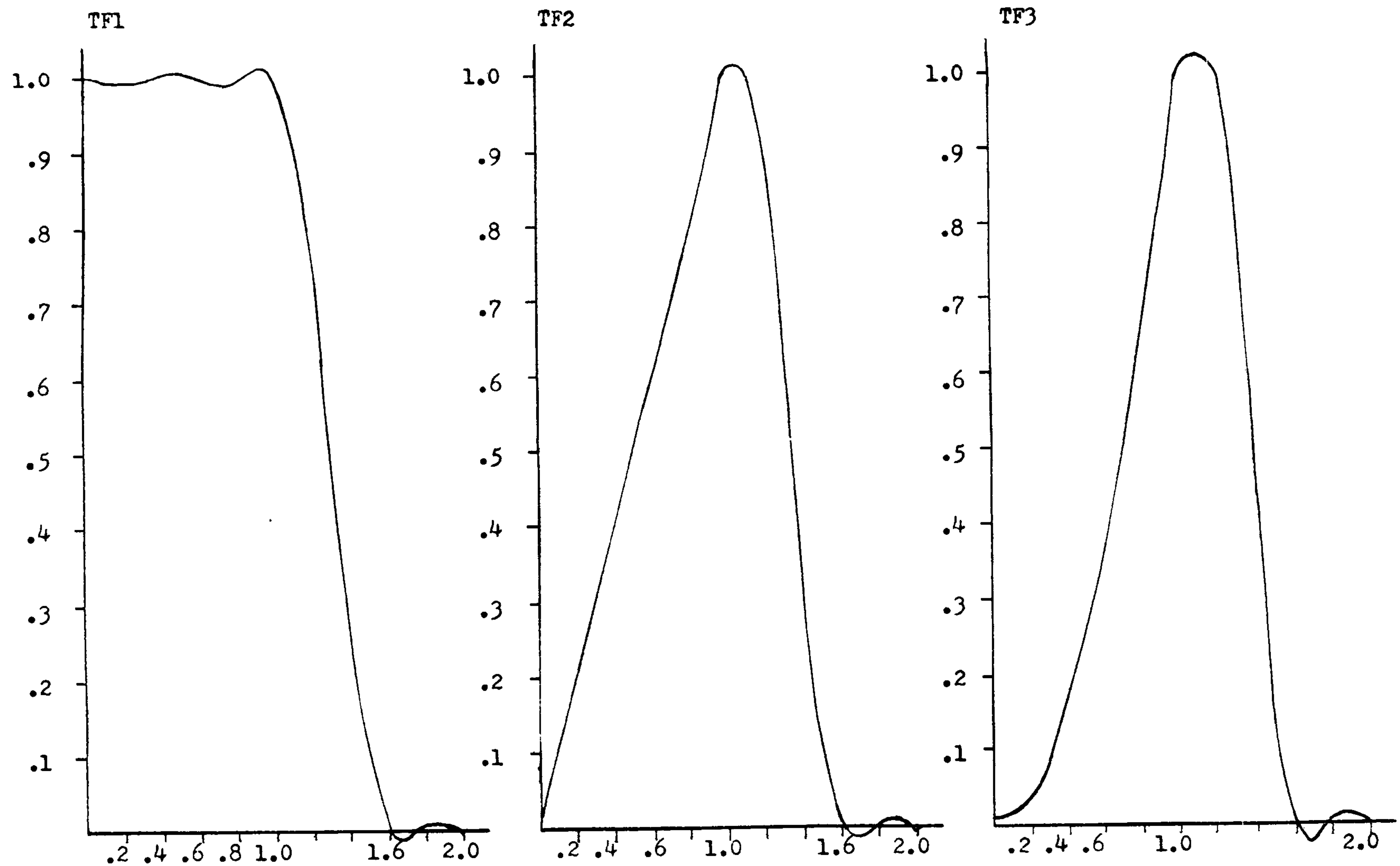


Figure 7.1. Run 1.20 transfer functions

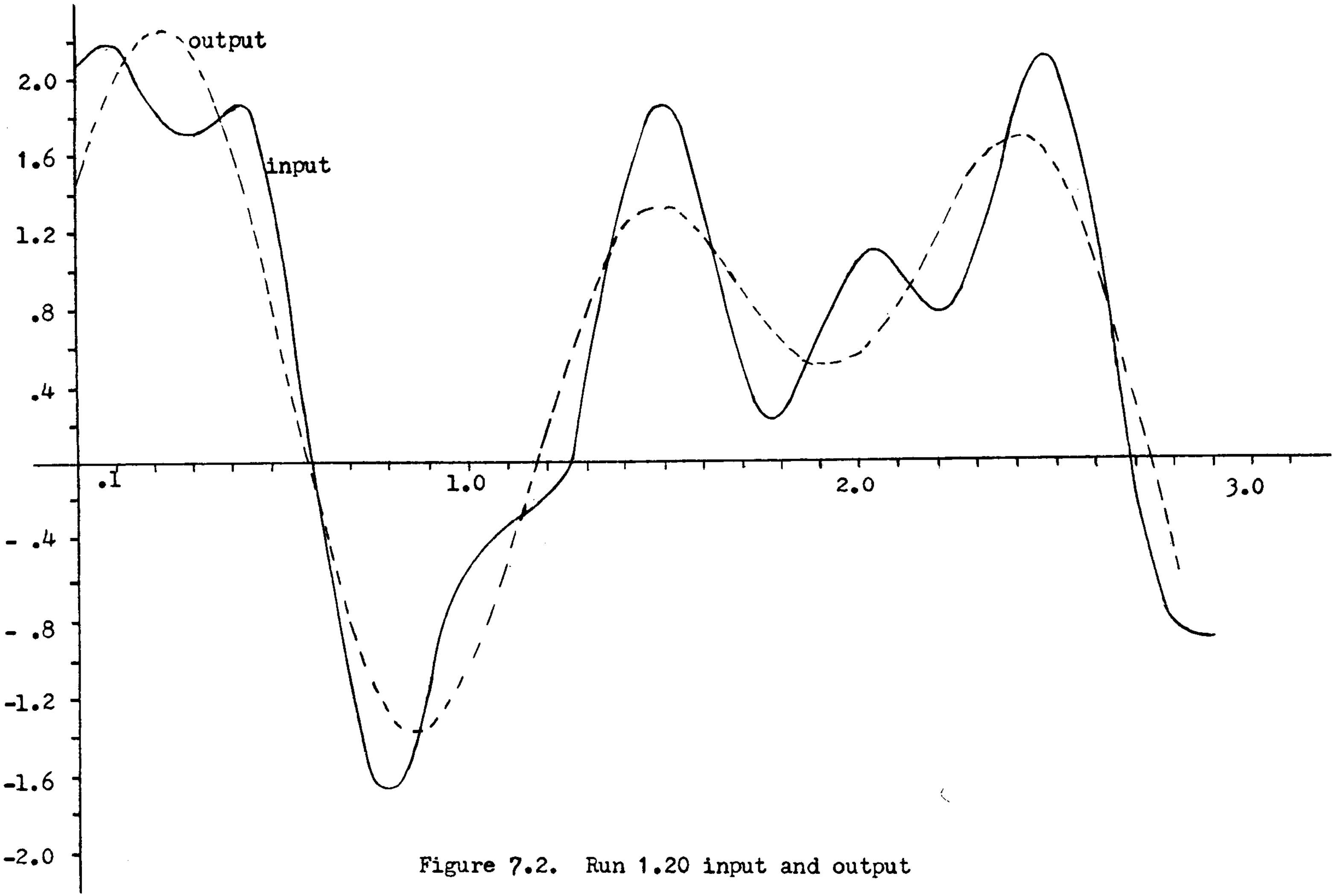


Figure 7.2. Run 1.20 input and output

7.4 Program and example for indefinite integration.

The following program was run with the input given by (7.7) with $A_1 = 1.5$, $A_2 = 2.0$, $A_3 = 1.5$, $A_4 = 0$, $f_1 = 0.7$, $f_2 = 0.9$, $f_3 = 2.0$, and $f_s = 10.0$. N = XN1 = 25. Also, 25 terms were used in computing the sine integral--which is too many terms for small values of the argument. For large values of the argument, the first terms of the series may become large enough to cause loss of significance, and computation of the sine integral should be approached with caution.

The other parameters chosen were $f_c = 1$, $\Delta f = .6$, Δf the inner and outer roll-off length. In terms of the frequency ratio, the input frequencies are .07, .09, and .2, and $r_c = .1$, $r_d = .06$, $r_T = .16$.

The notation chosen in the program is analogous to that of section 7.3. The results follow the program. In this case the weights are given along with the transfer function recovery, the desired output, and the actual output.

```
C   C  INDEFINITE INTEGRATION WITH SMOOTHING
       DIMENSION TERM1(50),TERM2(50),A(30),B(30)
       DIMENSION C(30),H(30),Z(101),ZI(40)
       P = 3.14159
       READ 1, XM1, XN1, TC, TD
     1 FORMAT (F10.0,F10.0,F10.0,F10.0)
       READ 2, T1, T2, T3, FS
     2 FORMAT (F10.0,F10.0,F10.0,F10.0)
       READ 3, A1, A2, A3
     3 FORMAT (F10.0,F10.0,F10.0)
       READ 4, B1, B2, B3
     4 FORMAT (F10.0,F10.0,F10.0)
       M1 = XM1
       N1 = XN1
       RT = TC + TD
C   C  SINE INTEGRAL
       TERM1(1)=1.
       TERM2(1)=1.
       DO 8  I = 1, N1
       X = I
       X1 = 2.*X*P*TC
       X2 = 2.*X*P*TD
       DO 5 K=1,M1
       Y = K
       J=K+1
       TERM1(J)=-X1*X1*TERM1(K)*(2.*Y-1.)/((2.*Y+1.)*(2.*Y+1.)*2.*Y)
     5 TERM2(J)=-X2*X2*TERM2(K)*(2.*Y-1.)/((2.*Y+1.)*(2.*Y+1.)*2.*Y)
       S1 = 0
       S2 = 0
       DO 6  J = 1, M1
       S1=S1+TERM1(J)
     6 S2=S2+TERM2(J)
C   C   WEIGHTS
       A(I)=2.*P*TD*(X2*S2-X1*S1)
       B(I) = (COSF(2.*P*X*TD)/X) - SINF(2.*P*X*TD)/(2.*P*TD*X*X)
       C(I) = (SINF(2.*P*X*RT) - SINF(2.*P*X*TC))/(2.*P*TC*X*X)
       H(I)=A(I)+B(I)+C(I)-TD*COSF(2.*P*X*TC)/(TC*X)
       PUNCH 7, X, H(I)
     7 FORMAT (F7.3, E20.8)
     8 CONTINUE
C   C  TRANSFER FUNCTION RECOVERY
       DO 11  K = 1, 51
       H1 = 0
       Y = K-1
       Y = Y * .01
       DO 9  I = 1, N1
       X = I
     9 H1 = H1 + 2.*H(I)*SINF(2.*X*P*Y)
       H1 = H1/(2.*P*P*TD*FS)
       Y1=Y*FS
       PUNCH 10,Y1,H1
    10 FORMAT (F7.3, E20.8)
    11 CONTINUE
C   C  INPUT DATA
       M3=N1+1
       M = 2*N1+40
```

```
      P2 = P*P
      DO 14  I = 1, M
      T = I-M3
      C1 = COSF(2.*P*T1*T)
      S = SINF(2.*P*T2*T)
      C3=COSF(2.*P*T3*T)
   14 Z(I)=A1*C1+A2*S+A3*C3
C  C  DESIRED OUTPUT
      DO 16  I = 1, 40
      T = I-1
      C1 = SINF(2.*P*T1*T)
      S = COSF(2.*P*T2*T)
      C3=SINF(2.*P*T3*T)
      ZI(I)=(1./(2.*P*FS))*((B1*C1/T1)-(B2*S/T2)+(B3*C3/T3))
      Y=T/FS
      PUNCH 15,Y,ZI(I)
   15 FORMAT (F7.3, E20.8)
   16 CONTINUE
C  C  ACTUAL OUTPUT
      DO 19  K = 1, 40
      M4 = K-1
      M5 = N1+1
      S1 = 0
      T = M4
      T = T/FS
      DO 17  I = 1, N1
      K1 = M5-I
      K2 = I + M4
      K3 = M5+M4+I
   17 S1 = S1 - H(K1)*Z(K2)+H(I)*Z(K3)
      S1 = S1/(2.*P2*TD*FS)
      PUNCH 18, T,. S1
   18 FORMAT (F7.3, E20.8)
   19 CONTINUE
      STOP
      END
```

Indefinite integration with smoothing. Weights and Transfer function.

n	Weights	f	Transfer function
		.000	.00000000E-99
1.000	-.21454526E-00	.100	-.48316595E-01
2.000	-.35924655E-00	.200	-.87799236E-01
3.000	-.39618853E-00	.300	-.12781742E-00
4.000	-.33380622E-00	.400	-.17900819E-00
5.000	-.21699964E-00	.500	-.22789864E-00
6.000	-.10052025E-00	.600	-.24757827E-00
7.000	-.22045076E-01	.700	-.23157831E-00
8.000	.10480938E-01	.800	-.20080206E-00
9.000	.13045094E-01	.900	-.17572434E-00
10.000	.85941740E-02	1.000	-.15644913E-00
11.000	.11353359E-01	1.100	-.13391492E-00
12.000	.21432966E-01	1.200	-.10680382E-00
13.000	.30236526E-01	1.300	-.79328906E-01
14.000	.30126059E-01	1.400	-.52227487E-01
15.000	.20557104E-01	1.500	-.25658076E-01
16.000	.73245800E-02	1.600	-.54776971E-02
17.000	-.28660550E-02	1.700	.22907104E-02
18.000	-.72928270E-02	1.800	.69471017E-03
19.000	-.78980180E-02	1.900	-.14145407E-02
20.000	-.81692720E-02	2.000	-.27708135E-03
21.000	-.99475460E-02	2.100	.99977962E-03
22.000	-.11789276E-01	2.200	.16964473E-03
23.000	-.11421037E-01	2.300	-.76567173E-03
24.000	-.62803660E-02	2.400	-.13772748E-03
25.000	-.95407800E-03	2.500	.60982346E-03
		2.600	.13198459E-03
		2.700	-.49169692E-03
		2.800	-.13306796E-03
		2.900	.39533037E-03
		3.000	.13152694E-03
		3.100	-.31636442E-03
		3.200	-.12515112E-03
		3.300	.25396192E-03
		3.400	.11672168E-03
		3.500	-.20549376E-03
		3.600	-.10984179E-03
		3.700	.16551916E-03
		3.800	.10457756E-03
		3.900	-.12943515E-03
		4.000	-.98196927E-04
		4.100	.95910881E-04
		4.200	.88133242E-04
		4.300	-.65782975E-04
		4.400	-.73275802E-04
		4.500	.40405606E-04
		4.600	.53352386E-04
		4.700	-.21199889E-04
		4.800	-.28416106E-04
		4.900	.10545189E-04
		5.000	.13537622E-08

Indefinite integration and smoothing. Desired output and actual output.

t	Desired output	t	Actual output
.000	-.35367793E-00	.000	-.35144868E-00
.100	-.15340979E-00	.100	-.14844095E-00
.200	.11219169E-00	.200	.11825555E-00
.300	.37465896E-00	.300	.38025760E-00
.400	.56044806E-00	.400	.56484080E-00
.500	.61228028E-00	.500	.61527389E-00
.600	.50686785E-00	.600	.50814904E-00
.700	.26352543E-00	.700	.26264061E-00
.800	-.59272709E-01	.800	-.62261615E-01
.900	-.37880758E-00	.900	-.38298996E-00
1.000	-.61048465E-00	1.000	-.61469266E-00
1.100	-.69133740E-00	1.100	-.69498835E-00
1.200	-.59788727E-00	1.200	-.60102638E-00
1.300	-.35313085E-00	1.300	-.35568815E-00
1.400	-.20540975E-01	1.400	-.21868438E-01
1.500	.31327197E-00	1.500	.31391495E-00
1.600	.56230139E-00	1.600	.56495094E-00
1.700	.66451015E-00	1.700	.66844175E-00
1.800	.59819582E-00	1.800	.60263503E-00
1.900	.38682170E-00	1.900	.39141123E-00
2.000	.91174926E-01	2.000	.95579163E-01
2.100	-.20860285E-00	2.100	-.20530551E-00
2.200	-.43570072E-00	2.200	-.43481667E-00
2.300	-.53740801E-00	2.300	-.53966433E-00
2.400	-.49810027E-00	2.400	-.50302054E-00
2.500	-.34105079E-00	2.500	-.34737171E-00
2.600	-.11908317E-00	2.600	-.12560173E-00
2.700	.10262141E-00	2.700	.96819607E-01
2.800	.26607147E-00	2.800	.26204345E-00
2.900	.33641861E-00	2.900	.33549022E-00
3.000	.30975637E-00	3.000	.31278276E-00
3.100	.21090859E-00	3.100	.21739711E-00
3.200	.82557013E-01	3.200	.90734710E-01
3.300	-.30313693E-01	3.300	-.22491099E-01
3.400	-.95377830E-01	3.400	-.89373466E-01
3.500	-.10249805E-00	3.500	-.99234314E-01
3.600	-.64953839E-01	3.600	-.65210438E-01
3.700	-.12357850E-01	3.700	-.16574201E-01
3.800	.21974700E-01	3.800	.14440005E-01
3.900	.14623757E-01	3.900	.57324858E-02

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APPENDIX A

CONSTRAINTS

In order to develop constraints on the weights h_k such that the recovered transfer function $\overline{H}$ has an exact fit at some specified frequency $\overline{r}$ we need to consider two separate cases. The first is

when H is of the form $H(r) = h_o + 2\sum_{n=1}^{N} h_n \cos 2\pi nr$, $r = \frac{f}{f_s} = \frac{\pi}{2\pi f_s}$.

The second is when H(r) is of the form $H(r) = 2i\sum_{n=1}^{N} h_n \sin 2\pi nr$.

A.1 Constraints at one point

Case I. Suppose $\overline{H}(r) = \overline{h}_o + 2\sum_{n=1}^{N} \overline{h}_n \cos 2\pi nr$,

then

$$\overline{H}'(r) = -4\pi\sum_{n=1}^{N} n\overline{h}_n \sin 2\pi nr.$$

We wish to impose the following constraints:

$$\overline{H}(\overline{r}) = F(\overline{r}),$$

$$\overline{H}'(\overline{r}) = F'(\overline{r}),$$

i.e.,

$$\overline{h}_o + 2\sum_{n=1}^{N} \overline{h}_n \cos 2\pi n\overline{r} - F(\overline{r}) = 0,$$

$$4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r}) = 0.$$

In order to minimize the error between H and $\bar{H}$ under the above constraints we define

$$R = \int_0^{\frac{1}{2}} [\bar{H}(r) - H(r)]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r})].$$

Since

$$\bar{h}_o = F(\bar{r}) - 2\sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r},$$

$$R = \int_0^{\frac{1}{2}} [F(\bar{r}) + 2\sum_{n=1}^{N} \bar{h}_n(\cos 2\pi nr - \cos 2\pi n\bar{r}) - h_o - 2\sum_{n=1}^{N} h_n \cos 2\pi nr]^2 dr$$

$$+ \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r})].$$

$$\frac{\partial R}{\partial \bar{h}_k} = 2\int_0^{\frac{1}{2}} [F(\bar{r}) + 2\sum_{n=1}^{N} \bar{h}_n(\cos 2\pi nr - \cos 2\pi n\bar{r}) - h_o - 2\sum_{n=1}^{N} h_n \cos 2\pi nr]$$

$$[\cos 2\pi kr - \cos 2\pi k\bar{r}]dr + \alpha[4\pi k \sin 2\pi k\bar{r}].$$

Let $\frac{\partial R}{\partial \bar{h}_k} = 0$, $k = 1, \ldots, N$.

$$-\frac{1}{2} F(\bar{r})\cos 2\pi k\bar{r} + \frac{\bar{h}_k}{2} + \sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r} \cos 2\pi k\bar{r} + \frac{h_o}{2}\cos 2\pi k\bar{r} - \frac{h_k}{2}$$

$$= -\alpha[\pi k \sin 2\pi k\bar{r}]$$

$$\frac{1}{2}(\bar{h}_k - h_k) + [\sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r} - \frac{F(\bar{r})}{2}]\cos 2\pi k\bar{r}$$

$$+ \frac{h_o}{2}\cos 2\pi k\bar{r} = - \alpha[\pi k \sin 2\pi k\bar{r}].$$

Let $\delta = (\bar{h}_o - h_o)$, then

$$(\bar{h}_k - h_k) = \delta \cos 2\pi k\bar{r} - 2\alpha[\pi k \sin 2\pi k\bar{r}]. \tag{A.1}$$

Multiply (A.1) by $(2 \cos 2\pi k\bar{r})$. Summing from 1 to N gives

$$2\sum_{k=1}^{N}(\bar{h}_k - h_k)\cos 2\pi k\bar{r} = 2\sum_{k=1}^{N}\cos^2 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N}\pi k \cos 2\pi k\bar{r} \sin 2\pi k\bar{r},$$

adding $(\bar{h}_o - h_o)$ to both sides gives

$$(\bar{h}_o - h_o) + 2\sum_{k=1}^{N}(\bar{h}_k - h_k)\cos 2\pi k\bar{r}$$

$$= \delta + 2\delta\sum_{k=1}^{N}\cos^2 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N}\pi k \cos 2\pi k\bar{r} \sin 2\pi k\bar{r}.$$

Let

$$\Delta_1 = h_o + 2\sum_{n=1}^{N} h_n \cos 2\pi n\bar{r} - F(\bar{r}).$$

Hence

$$\Delta_1 = (h_o - \bar{h}_o) + 2\sum_{n=1}^{N}(h_n - \bar{h}_n)\cos 2\pi n\bar{r},$$

so

$$\Delta = 4\alpha\sum_{n=1}^{N}\pi n \cos 2\pi n\bar{r} \sin 2\pi n\bar{r} - \delta - 2\delta\sum_{n=1}^{N}\cos^2 2\pi n\bar{r}.$$

Now multiply (A.1) by $2k\sin 2\pi k\bar{r}$. Summing from 1 to N gives

$$2\sum_{k=1}^{N} k(\bar{h}_k - h_k)\sin 2\pi k\bar{r} = 2\,\delta\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N} \pi k^2 \sin^2 2\pi k\bar{r}.$$

Let

$$\Delta_2 = -4\pi\sum_{n=1}^{N} nh_n \sin 2\pi n\bar{r} - F'(\bar{r}).$$

Hence

$$\Delta_2 = 4\pi\sum_{n=1}^{N} n(\bar{h}_n - h_n)\sin 2\pi n\bar{r}.$$

So

$$\Delta_2 = 4\pi\,\delta\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r} - 8\pi^2\alpha\sum_{k=1}^{N} k^2\sin^2 2\pi k\bar{r}.$$

Let $Q_1 = 2\sum_{k=1}^{N}\cos^2 2\pi k\bar{r},$

$$Q_2 = 4\pi\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r},$$

$$Q_3 = 8\pi^2\sum_{k=1}^{N} k^2\sin^2 2\pi k\bar{r}.$$

Then

$$\Delta_1 = Q_2\alpha - (1+Q_1)\,\delta\,,$$

and

$$\Delta_2 = -Q_3\alpha + Q_2\,\delta\,.$$

Solving we find that

$$\delta = \frac{\Delta_1 Q_3 + \Delta_2 Q_2}{Q_2^2 - (1+Q_1)Q_3}, \tag{A.2}$$

$$\alpha = \frac{\Delta_1 Q_2 + \Delta_2 (1+Q_1)}{Q_2^2 - (1+Q_1)Q_3}. \tag{A.3}$$

Therefore the constrained weights are

$$\bar{h}_o = h_o + \delta \ ,$$

$$\bar{h}_k = h_k + \delta \ \cos 2\pi k\bar{r} - \alpha 2\pi k \sin 2\pi k\bar{r}, \quad k \geq 1,$$

where δ and α are as defined in (A.2) and (A.3).

<u>Case II.</u> Suppose $\bar{H}(r) = 2i \sum_{n=1}^{N} \bar{h}_n \sin 2\pi nr,$

then

$$\bar{H}'(r) = 4\pi i \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi nr.$$

We wish to impose the following constraints

$$\bar{H}(\bar{r}) = F(\bar{r}),$$

$$\bar{H}'(\bar{r}) = F'(\bar{r}),$$

i.e.,

$$2 \sum_{n=1}^{N} \bar{h}_n \sin 2\pi n\bar{r} - \frac{F(\bar{r})}{i} = 0,$$

and

$$4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F'(\bar{r})}{i} = 0.$$

In order to minimize the error between H and $\bar{H}$ under the above conditions we define

$$R = \int_0^{\frac{1}{2}} [\bar{H}(r) - H(r)]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F'(\bar{r})}{i}].$$

Since

$$\bar{h}_1 = \frac{\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}}{2\sin 2\pi\bar{r}},$$

$$R = \int_0^{\frac{1}{2}} [\frac{\sin 2\pi r}{\sin 2\pi\bar{r}} [\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}] + 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi nr$$

$$- 2\sum_{n=1}^{N} h_n \sin 2\pi nr]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F(\bar{r})}{i}]$$

$$\frac{\partial R}{\partial \bar{h}_k} = 2\int_0^{\frac{1}{2}} [\frac{\sin 2\pi r}{\sin 2\pi\bar{r}}[\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}] + 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi nr$$

$$-2\sum_{n=1}^{N} h_n \sin 2\pi nr][\frac{-2\sin 2\pi r \sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} + 2\sin 2\pi kr]dr + 4\pi\alpha k\cos 2\pi k\bar{r}.$$

Let $\frac{\partial R}{\partial \bar{h}_k} = 0$, $k = 2, \ldots, N$.

Now $$\int_0^{\frac{1}{2}} \frac{\sin 2\pi r}{\sin 2\pi\bar{r}} [\frac{F(r)}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}][\sin 2\pi kr - \frac{\sin 2\pi r \sin 2\pi kr}{\sin 2\pi\bar{r}}]dr$$

$$= - \frac{F(\bar{r})}{4i} \frac{\sin 2\pi k\bar{r}}{\sin^2 2\pi\bar{r}} + \frac{\sin 2\pi k\bar{r}}{2\sin^2 2\pi\bar{r}} \sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}$$

Also $\int_0^{\frac{1}{2}} \sum_{n=2}^{N} [\bar{h}_n - h_n] \sin 2\pi n r [\sin 2\pi k r - \frac{\sin 2\pi r \sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}}] dr = \frac{1}{4}[\bar{h}_k - h_k],$

$$\int_0^{\frac{1}{2}} [h_1 \sin^2 2\pi r] \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}} dr = \frac{h_1}{4} \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}} .$$

Hence

$$\frac{1}{4} [2 \sum_{n=2}^{N} \bar{h}_n \sin 2\pi n \bar{r} - \frac{F(\bar{r})}{i}] \frac{\sin 2\pi k \bar{r}}{\sin^2 2\pi \bar{r}} + \frac{1}{4} [\bar{h}_k - h_k] + \frac{h_1}{4} \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}}$$

$$= - \frac{\alpha}{4} \pi k \cos 2\pi k \bar{r}$$

$$(h_1 - \bar{h}_1) \frac{\sin 2\pi \bar{r} k}{\sin 2\pi \bar{r}} + (\bar{h}_k - h_k) = - \alpha \pi k \cos 2\pi k \bar{r}.$$

Let $\delta = \bar{h}_1 - h_1$, then

$$\bar{h}_k - h_k = \delta \frac{\sin 2\pi \bar{r} k}{\sin 2\pi \bar{r}} - \alpha \pi k \cos 2\pi k \bar{r}. \tag{A.4}$$

Multiplying (A.4) by $2 \sin 2\pi k \bar{r}$ and summing from 2 to N gives

$$2 \sum_{k=2}^{N} (\bar{h}_k - h_k) \sin 2\pi k \bar{r} = \delta\, 2 \sum_{k=2}^{N} \frac{\sin^2 2\pi k \bar{r}}{\sin 2\pi \bar{r}} - \alpha 2 \sum_{k=2}^{N} k \cos 2\pi k \bar{r} \sin 2\pi k \bar{r}.$$

Adding $2(\bar{h}_1 - h_1) \sin 2\pi \bar{r}$ to both sides yields

$$2 \sum_{k=1}^{N} (\bar{h}_k - h_k) \sin 2\pi k \bar{r} = 2\, \delta \sum_{k=1}^{N} \frac{\sin^2 2\pi k \bar{r}}{\sin 2\pi \bar{r}} - 2\alpha \sum_{k=2}^{N} k \cos 2\pi k \bar{r} \sin 2\pi k \bar{r}.$$

Let $\Delta_1 = 2 \sum_{k=1}^{N} h_k \sin 2\pi k \bar{r} - F(\bar{r})$.

Since

$$F(\bar{r}) = 2\sum_{k=1}^{N} \bar{h}_k \sin 2\pi k\bar{r},$$

$$\Delta_1 = 2\sum_{k=1}^{N} (h_k - \bar{h}_k) \sin 2\pi k\bar{r},$$

or

$$\Delta_1 = 2\alpha\sum_{k=2}^{N} k\cos 2\pi k\bar{r} \sin 2\pi k\bar{r} - 2\,\delta\sum_{k=1}^{N} \frac{\sin^2 2\pi k\bar{r}}{\sin 2\pi\bar{r}} .$$

Multiplying (A.4) by $4\pi k\cos 2\pi k\bar{r}$ and summing from 2 to N gives

$$4\pi\sum_{k=1}^{N} (\bar{h}_k - h_k) k\cos 2\pi k\bar{r} = \delta\, 4\pi\sum_{k=1}^{N} \frac{k\sin 2\pi k\bar{r} \cos 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\sum_{k=2}^{N} \pi^2 k^2 \cos^2 2\pi k\bar{r},$$

adding $4\pi(\bar{h}_1 - h_1)\cos 2\pi\bar{r}$ to both sides of the above equation gives

$$4\pi\sum_{k=1}^{N} (\bar{h}_k - h_k) k\cos 2\pi k\bar{r} = \delta\, 4\pi\sum_{k=1}^{N} \frac{k\cos 2\pi k\bar{r} \sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\sum_{k=2}^{N} \pi^2 k^2 \cos^2 2\pi k\bar{r}.$$

Let $\Delta_2 = 4\pi\sum_{k=1}^{N} k h_k \cos 2\pi k\bar{r} - F'(\bar{r}).$

Since $F'(\bar{r}) = 4\pi\sum_{k=1}^{N} k\bar{h}_k \cos 2\pi k\bar{r}.$

$$\Delta_2 = 4\pi\sum_{k=1}^{N} (h_k - \bar{h}_k) k\cos 2\pi k\bar{r}.$$

Hence

$$\Delta_2 = 4\pi\,\delta\sum_{k=1}^{N} \frac{k\cos 2\pi k\bar{r} \sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\pi^2\sum_{k=2}^{N} k^2 \cos^2 2\pi k\bar{r}.$$

Let

$$Q_1 = 2\sum_{k=1}^{N} \frac{\sin^2 2\pi k\bar{r}}{\sin 2\pi\bar{r}},$$

$$Q_2 = 2\sum_{k=1}^{N} k\cos 2\pi k\bar{r} \sin 2\pi k\bar{r},$$

$$Q_3 = \frac{2\pi Q_2}{\sin 2\pi\bar{r}}$$

$$Q_4 = 4\pi^2 \sum_{k=1}^{N} k^2\cos^2 2\pi k\bar{r}.$$

Then

$$\Delta_1 = \alpha(Q_2 - \cos 2\pi\bar{r} \sin 2\pi\bar{r}) - \delta Q_2,$$

and

$$\Delta_2 = -\alpha(Q_4 - 4\pi^2 \cos^2 2\pi\bar{r}) + \delta Q_3.$$

Solving for δ and α we find that

$$\delta = \frac{\Delta_1(Q_4-4\pi^2\cos^2 2\pi\bar{r}) + \Delta_2(Q_2-\cos 2\pi\bar{r} \sin 2\pi\bar{r})}{Q_3(Q_2-\cos 2\pi\bar{r} \sin 2\pi\bar{r}) - Q_2(Q_4-4\pi^2 \cos^2 2\pi\bar{r})} \tag{A.5}$$

$$\alpha = \frac{\Delta_1 Q_3 - \Delta_2 Q_2}{Q_3(Q_2-\cos 2\pi\bar{r} \sin 2\pi\bar{r}) - Q_2(Q_4-4\pi^2\cos^2 2\pi\bar{r}))}. \tag{A.6}$$

Therefore the constrained weights are

$$\bar{h}_1 = h_1 + \delta$$

$$\bar{h}_k = h_k + \delta \frac{\sin 2\pi\bar{r}k}{\sin 2\pi\bar{r}} - \alpha\pi k \cos 2\pi k\bar{r}, \quad k \geq 2.$$

APPENDIX B

ROMBERG'S METHOD OF NUMERICAL INTEGRATION

Suppose we wish to calculate $J = \int_a^b g(x)dx$. First we define

$$T_{o,k} = 2^{-k}\left[\frac{1}{2} g(a) + \sum_{n=1}^{2^k-1} g\left(a+\frac{b-a}{2^k} n\right) + \frac{1}{2} g(b)\right],$$

which are the results of applying the trapezoidal rule with a partitioning of the interval [a,b] into 2^k parts. With these values we form

$$T_{m,k} = \frac{4^m T_{m-1,\ k+1} - T_{m-1,k}}{4^m - 1},$$

m=1,2, ..., k=0, 1,2, ..., and arrange them in the triangular table

$$\begin{array}{llll}
T_{o,o} & & & \\
T_{o,1} & T_{1,o} & & \\
T_{o,2} & T_{1,1} & T_{2,o} & \\
T_{o,3} & T_{1,2} & T_{2,1} & T_{3,o} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array}$$

Then $J \doteq (b-a)\, T_{m,o}$. See [13].

If the interval [a,b] is divided into 2^k subintervals, each of length h, it has been shown that $J - (b-a)T_{m,o} = O(h^{2k+2})$.

We note that, assuming that a sufficiently large number of functional values are available in the interval, a very close approximation of the integral can be obtained.

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